

Question 7

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

An unknown signal belongs to the family $f(t) = e^{-at} \cos bt$, with $a \geq 0$ and $b > 0$. Its transform satisfies the exact observations

$$\lim_{s \rightarrow \infty} s[sF(s) - 1] = -2, \quad \lim_{s \rightarrow \infty} s^3[F(s) - 1/s + 2/s^2] = -5.$$

Tasks

1. Derive the transform in terms of a, b and expand it through the term in s^{-3} .
2. Recover a, b from the observations and state the resulting transform with its convergence interval.
3. Compute $f(0)$, $f'(0)$ and $f''(0)$ directly. Explain how the expansion coefficients encode these initial quantities.
4. For $q(t) = t^3 e^{-t}$, compute its transform and show that $f + \lambda q$ has the same two observed limits for every real λ . Explain the limit of the parameter recovery claim.

Question 7 – Solution

Strategy. Large transform parameters emphasize early time. Within a specified family, the first few coefficients can determine parameters, but they do not determine an arbitrary signal.

Step 1: Transform and expand. Exponential shifting in the cosine pair gives

$$F(s) = \frac{s+a}{(s+a)^2 + b^2}, \quad s > -a.$$

Expanding this rational expression for large positive s yields

$$F(s) = \frac{1}{s} - \frac{a}{s^2} + \frac{a^2 - b^2}{s^3} + O(s^{-4}).$$

For example, factor s^2 from the denominator and expand its reciprocal through order s^{-2} before multiplying by $1/s + a/s^2$.

Step 2: Recover the parameters. The first limit equals $-a$, so $a = 2$. With this value, the second equals $a^2 - b^2 = -5$, giving $b^2 = 9$. Since $b > 0$,

$$a = 2, \quad b = 3, \quad F(s) = \frac{s+2}{(s+2)^2 + 9}, \quad s > -2.$$

The boundary has an undamped cosine integral that does not converge; smaller parameters produce growing oscillations. Thus the interval is exact.

Step 3: Read the initial information. Direct differentiation gives

$$f(0) = 1, \quad f'(0) = -a = -2, \quad f''(0) = a^2 - b^2 = -5.$$

These are precisely the coefficients of $1/s$, $1/s^2$ and $1/s^3$ in the expansion. Repeated integration by parts explains the same pattern when the needed derivative bounds hold.

Step 4: Show what the measurements cannot identify. Three integrations by parts give $\mathcal{L}\{q\}(s) = 6/(s+1)^4$ for $s > -1$. Thus $f + \lambda q$ has transform $F(s) + 6\lambda/(s+1)^4$ on their common domain. The added term is $O(s^{-4})$, so it changes neither observed limit. Also $q(0) = q'(0) = q''(0) = 0$. Distinct λ therefore give different signals with the same recorded information. The parameters are uniquely recovered only within the stated exponential-cosine family.

