

Question 6

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

For $t > 0$, define

$$h(t) = \frac{e^{-t} - e^{-4t}}{t},$$

and extend it continuously to $t = 0$. This example concerns division by time; do not assume a transform rule for dividing an arbitrary function by t .

Tasks

1. Find $h(0)$ and express $h(t)$ as an integral of e^{-ut} over a finite interval in u .
2. Use that representation to derive the transform $H(s)$ and justify the order of integration.
3. Find the exact real convergence interval. Explain why transforming e^{-t}/t and e^{-4t}/t separately is invalid.
4. Find $H'(s)$ and $\lim_{s \rightarrow \infty} sH(s)$. Relate them to positivity of h and its value at zero.

Question 6 – Solution

Strategy. Cancellation removes the origin singularity. Preserve that cancellation by integrating over an exponential parameter.

Step 1: Remove the apparent singularity. Expansion at zero, or l'Hopital's rule, gives $\boxed{h(0) = 3}$. Direct integration in u gives

$$h(t) = \int_1^4 e^{-ut} du,$$

including at $t = 0$ by continuity. In particular, h is positive and bounded above by three.

Step 2: Derive the logarithmic transform. For $s > -1$, $s + u > 0$ throughout $[1, 4]$. The absolute double integral is finite, as seen by integrating first in t :

$$H(s) = \int_1^4 \int_0^\infty e^{-(s+u)t} dt du = \int_1^4 \frac{du}{s+u} = \boxed{\ln \frac{s+4}{s+1}}, \quad s > -1.$$

The finite value justifies interchange; both logarithm arguments are positive on this domain.

Step 3: Check the domain and invalid separation. At $s = -1$, the integrand is $(1 - e^{-3t})/t$, whose tail is at least $1/(2t)$ for sufficiently large t . It diverges. For smaller s , a larger positive exponential weight makes divergence persist. Thus $s > -1$ is the full real convergence interval. Each separate term e^{-t}/t or e^{-4t}/t has a nonintegrable origin singularity for every s . Their divergent transforms cannot be subtracted; it is the difference of the functions that is transformable.

Step 4: Interpret derivative and initial scale. Differentiating the logarithm gives

$$\boxed{H'(s) = \frac{1}{s+4} - \frac{1}{s+1} = -\frac{3}{(s+1)(s+4)} < 0.}$$

This agrees with $H'(s) = -\int_0^\infty te^{-st}h(t) dt$, since $h > 0$. Also $H(s) = \ln[1 + 3/(s+1)]$, so $\ln(1+x)/x \rightarrow 1$ implies $\boxed{sH(s) \rightarrow 3 = h(0)}$. The graph displays the continuous signal; its apparent division-by-zero point has been correctly filled in.

