

Question 5

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let $f(t) = e^{-t}(1+t)$. A student claims that the second derivative has transform $s^2 F(s)$, without any initial terms.

Tasks

1. Derive the first- and second-derivative transform formulas using integration by parts on a finite interval before taking the limit.
2. Compute $F(s)$, $f'(t)$ and $f''(t)$, and obtain their transforms explicitly.
3. Identify the omitted term in the student's claim and verify the corrected result independently from the expression for f'' .
4. Evaluate the transforms of f' and f'' at $s = 0$ and check them using endpoint differences. Explain how a nonzero second derivative can have zero total signed integral.

Question 5 – Solution

Strategy. Differentiation in time creates boundary terms. Keep them until the initial data and exponential decay have been used.

Step 1: Retain the boundaries. Integration by parts on $[0, R]$ gives

$$\int_0^R e^{-st} f'(t) dt = e^{-sR} f(R) - f(0) + s \int_0^R e^{-st} f(t) dt.$$

Repeating for f'' and then taking $R \rightarrow \infty$ yields

$$\mathcal{L}\{f'\} = sF - f(0), \quad \mathcal{L}\{f''\} = s^2F - sf(0) - f'(0),$$

provided the upper boundary terms vanish. Here they do for every $s > -1$, since f, f' are polynomials times e^{-t} .

Step 2: Compute the functions and transforms. Using elementary exponential integrals,

$$F(s) = \frac{1}{s+1} + \frac{1}{(s+1)^2} = \frac{s+2}{(s+1)^2}.$$

Direct differentiation gives $f' = -te^{-t}$ and $f'' = (t-1)e^{-t}$, with $f(0) = 1$ and $f'(0) = 0$. Hence

$$\boxed{\mathcal{L}\{f'\} = -\frac{1}{(s+1)^2}, \quad \mathcal{L}\{f''\} = -\frac{s}{(s+1)^2}, \quad s > -1.}$$

These intervals are exact: at or below -1 , the resulting polynomial or growing exponential tails do not have convergent improper integrals.

Step 3: Repair and check the claim. The omitted initial contribution is $-sf(0) - f'(0) = -s$. Thus the corrected expression is $s^2F - s = -s/(s+1)^2$. Independently, transforming $(t-1)e^{-t}$ gives

$$\frac{1}{(s+1)^2} - \frac{1}{s+1} = -\frac{s}{(s+1)^2}.$$

This agreement verifies both the boundary terms and the algebra.

Step 4: Check zero-parameter integrals. At zero, the first-derivative transform equals -1 , consistent with $\int_0^\infty f' = f(\infty) - f(0) = 0 - 1$. The second-derivative transform is zero, consistent with $\int_0^\infty f'' = f'(\infty) - f'(0) = 0$. The function f'' is negative before $t = 1$ and positive afterward; its two signed contributions cancel. Zero integral is not the same as a zero function.