

Question 4

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let $g(t) = e^{-t} \cos t$ and $y(t) = \int_0^t g(u) du$. The transform of g is $G(s) = (s+1)/[(s+1)^2 + 1]$ for $s > -1$.

Tasks

1. Derive the transform rule for y by integration by parts, showing where the zero initial value and the boundary at infinity enter.
2. Find an elementary expression for $y(t)$, its transform $Y(s)$, and the exact real convergence interval of Y .
3. Set $z(t) = y(t) - 1/2$. Find its transform and convergence interval, including its value at $s = 0$. Explain why cancellation can enlarge the interval.
4. Find the first maximum of y and its height. Does integration of this exponentially decaying input produce an exponentially decaying or monotone cumulative response?

Question 4 – Solution

Strategy. A cumulative integral may retain a nonzero constant even when its input decays. Its transform domain must be checked anew.

Step 1: Derive the cumulative rule. Here $y(0) = 0$, $y' = g$, and y is bounded because g is absolutely integrable. For $s > 0$, integration by parts gives

$$G(s) = [e^{-st}y(t)]_0^\infty + s \int_0^\infty e^{-st}y(t) dt = sY(s).$$

The initial term is zero and the upper term vanishes because y is bounded. Thus $Y = G/s$ on this half-line.

Step 2: Determine the response and its domain. An antiderivative gives

$$y(t) = \frac{1 + e^{-t}(\sin t - \cos t)}{2}, \quad Y(s) = \frac{s+1}{s[(s+1)^2 + 1]}, \quad s > 0.$$

The formula satisfies $y(0) = 0$ and $y' = g$. Since $y(t) \rightarrow 1/2$, it is eventually bounded below by a positive constant. Its transform therefore diverges for every $s \leq 0$. The cumulative response has a smaller convergence interval than the input.

Step 3: Remove the persistent constant. Now $z = e^{-t}(\sin t - \cos t)/2$. Transforming these two terms directly gives

$$Z(s) = -\frac{s}{2[(s+1)^2 + 1]}, \quad s > -1.$$

It converges absolutely there; at and below -1 the nondecaying or growing trigonometric tail prevents ordinary convergence. In particular, $Z(0) = 0$: its signed areas cancel. The identity $Z = Y - 1/(2s)$ is initially valid only for $s > 0$. The wider domain follows from the actual centered function, not by subtracting divergent integrals at $s = 0$.

Step 4: Locate the overshoot. Since $y' = e^{-t} \cos t$, the first positive turning time is $t = \pi/2$, with derivative changing from positive to negative. Its height is

$$y(\pi/2) = \frac{1}{2}(1 + e^{-\pi/2}) > \frac{1}{2}.$$

The response overshoots and oscillates about its limiting constant, so it is neither monotone nor decaying to zero. Its centered part does decay exponentially.

