

Question 3

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let $g(t) = e^{-t} \sin 2t$, with $G(s) = 2/[(s+1)^2 + 4]$ for $s > -1$. Define the compressed signals $f(t) = g(3t)$ and $h(t) = 3g(3t)$.

Tasks

1. Derive the time-scaling identity for $g(at)$, $a > 0$, by substitution in the defining integral. Track the convergence parameter.
2. Find the transforms of f and h , and their full real convergence intervals. Explain why replacing $G(s)$ by $G(3s)$ gives the wrong scaling.
3. Compute the signed total integrals of g, f, h . Which compression preserves the total integral?
4. Use parameter differentiation to derive the first-moment identity. Compute $\int_0^\infty t f(t) dt$ and $\int_0^\infty t h(t) dt$. Explain their different scaling factors.

Question 3 – Solution

Strategy. Time compression changes both the exponential weight and the integration measure; amplitude compensation affects area separately.

Step 1: Derive the scaling rule. For $a > 0$, substituting $u = at$ gives

$$\int_0^\infty e^{-st} g(at) dt = \frac{1}{a} \int_0^\infty e^{-(s/a)u} g(u) du = \boxed{\frac{1}{a} G(s/a)}.$$

The effective parameter is s/a , so $s/a > -1$ here. Both the reciprocal parameter scaling and the factor $1/a$ are essential.

Step 2: Apply the rule to both signals. For $a = 3$,

$$\boxed{F(s) = \frac{6}{(s+3)^2 + 36}, \quad H(s) = \frac{18}{(s+3)^2 + 36}, \quad s > -3.}$$

These are the transforms of $e^{-3t} \sin 6t$ and three times that signal. At the boundary the undamped sine integral fails to converge; below it the exponentially growing sine fails the Cauchy criterion on positive half-wave subintervals. Thus $s > -3$ is exact. $G(3s)$ would stretch the parameter in the opposite direction and omit the measure factor.

Step 3: Compare signed areas. Since zero lies in the convergence intervals, substitution of $s = 0$ gives

$$\int_0^\infty g = \frac{2}{5}, \quad \int_0^\infty f = \frac{2}{15}, \quad \int_0^\infty h = \frac{2}{5}.$$

Compression alone reduces signed area by three; multiplying the amplitude by three restores it. These are signed integrals, not integrals of absolute value.

Step 4: Track first moments. Exponential decay justifies differentiation near $s = 0$, so $G'(s) = -\int_0^\infty te^{-st} g(t) dt$. As $G'(0) = -4/25$, the original first moment is $4/25$. Differentiating $F(s) = G(s/3)/3$ and $H(s) = G(s/3)$ yields

$$\boxed{\int_0^\infty tf(t) dt = \frac{4}{225}, \quad \int_0^\infty th(t) dt = \frac{4}{75}.$$

For f , one factor $1/3$ comes from dt and another from $t = u/3$. The amplitude factor in h cancels only one of them.