

Question 1

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let $f(t) = \sin^2(3t)$. A student proposes

$$\mathcal{L}\{\sin^2(3t)\}(s) = [\mathcal{L}\{\sin 3t\}(s)]^2.$$

You may use $\mathcal{L}\{1\} = 1/s$, $\mathcal{L}\{\cos bt\} = s/(s^2 + b^2)$ and $\mathcal{L}\{\sin bt\} = b/(s^2 + b^2)$ for $s > 0$.

Tasks

1. Find the correct transform using a trigonometric identity and linearity.
2. Determine the full real convergence set and compare the correct result with the student's proposed expression.
3. Use $|\sin x| \leq |x|$ to prove a direct upper bound for the transform when $s > 0$.
4. Use $u = st$ in the defining integral to derive $\lim_{s \rightarrow \infty} s^3 F(s)$ independently of the rational formula. Explain how this limit diagnoses the proposed product rule.

Question 1 – Solution

Strategy. Linearity applies to sums and scalar multiples, not to pointwise products. A large-parameter check can expose the distinction.

Step 1: Rewrite the function before transforming. Since $\sin^2(3t) = (1 - \cos 6t)/2$, linearity gives

$$F(s) = \frac{1}{2s} - \frac{s}{2(s^2 + 36)} = \boxed{\frac{18}{s(s^2 + 36)}}, \quad s > 0.$$

Step 2: Check convergence and the proposed rule. For $s > 0$, boundedness of f gives absolute convergence. For $s \leq 0$, the weight is at least one and $\sin^2(3t)$ has positive integral on each period, so the nonnegative improper integral diverges. Thus the real convergence set is exactly $(0, \infty)$. The proposed expression is

$$\frac{9}{(s^2 + 9)^2},$$

which is not the correct transform. For example, at $s = 3$ the correct value is $2/15$, while the proposed value is $1/36$. Squaring a time function is not transformed by squaring its transform.

Step 3: Obtain a bound from the integral. The inequality $0 \leq \sin^2(3t) \leq 9t^2$ yields

$$0 < F(s) \leq 9 \int_0^\infty t^2 e^{-st} dt = \boxed{18/s^3}.$$

The last integral is $2/s^3$, by two integrations by parts. Its boundary terms vanish for $s > 0$.

Step 4: Recover the leading scale independently. With $u = st$,

$$s^3 F(s) = \int_0^\infty e^{-u} s^2 \sin^2(3u/s) du.$$

The integrand tends to $9u^2 e^{-u}$ and is bounded by that integrable function. Dominated convergence therefore gives $\boxed{s^3 F(s) \rightarrow 18}$. This reflects the local behavior $\sin^2(3t) \sim 9t^2$. In contrast, multiplying the proposed expression by s^3 gives a limit of zero. The proposed rule even predicts the wrong leading power of $1/s$.