

Question 10

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

Use the chapter's table and operational rules to solve

$$y'' + 2y' + y = H_1(t)(t-1)e^{-(t-1)} + 2\delta(t-3), \quad y(0) = 1, \quad y'(0) = -1.$$

Ordinary force impulses preserve displacement and may jump velocity. All figures belong on the solution page.

Tasks

1. Find $Y(s)$, carefully retaining both initial terms and the two different forcing operations.
2. Invert the transform and give the response on the three intervals separated by 1 and 3.
3. Verify the initial state, the equation between switches, and the one-sided displacement and velocity changes at each switch.
4. Compute the total response area in two ways and determine the exact real transform domain. Explain why this finite list of table operations gives a unique solution.

Question 10 – Solution

Strategy. Separate the initial response, delayed ordinary forcing and impulse response before inversion.

Step 1: Transform with all data. The left side transforms to $(s^2Y - s + 1) + 2(sY - 1) + Y = (s + 1)^2Y - s - 1$. Consequently

$$Y = \frac{1}{s + 1} + \frac{e^{-s}}{(s + 1)^4} + \frac{2e^{-3s}}{(s + 1)^2}.$$

The ordinary forcing has transform $e^{-s}/(s + 1)^2$; the impulse has transform $2e^{-3s}$.

Step 2: Invert and make the intervals explicit. Define $v(\tau) = \tau^3 e^{-\tau}/6$ and $h(\tau) = \tau e^{-\tau}$ for $\tau \geq 0$. The factorial in the fourth-power table row gives

$$y(t) = \begin{cases} e^{-t}, & 0 \leq t < 1, \\ e^{-t} + v(t - 1), & 1 \leq t < 3, \\ e^{-t} + v(t - 1) + 2h(t - 3), & t \geq 3. \end{cases}$$

Step 3: Verify each kind of activation. The initial branch gives $(y(0), y'(0)) = (1, -1)$. For $D = d/d\tau$ and any polynomial P , $(D + 1)^2[e^{-\tau}P(\tau)] = e^{-\tau}P''(\tau)$. Thus v produces $\tau e^{-\tau}$ and h produces zero between switches. At 1, $v(0) = v'(0) = 0$, so displacement and velocity remain continuous. At 3, $h(0) = 0$, $h'(0) = 1$, so $[y]_3 = 0$ and $[y']_3 = 2$. Here $[z]_a = z(a^+) - z(a^-)$. These are exactly the required jump conditions.

Step 4: Check area, convergence and uniqueness. Direct areas are 1, $3!/6 = 1$, and 2, giving $\int_0^\infty y = 4$. Independently, integrating the equation, including the velocity jump, gives $1 - 2 + \int y = 1 + 2$, again yielding $\int y = 4$. The tail is a nonzero cubic polynomial times e^{-t} , so the exact real domain is $s > -1$. Linear IVP uniqueness on each ordinary interval, together with the forced matching and jump conditions, fixes the whole solution.

