

Question 9

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A measured response is known to lie in the three-mode family

$$f(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}.$$

The available data are $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 2$.

Tasks

1. Recover the three coefficients and prove uniqueness within the stated family.
2. Construct $F(s)$ by table lookup and simplify it to one rational expression. Check its leading large- s term against the data.
3. Prove that the recovered function is positive for every $t > 0$ despite a negative coefficient. Find its unique global maximum.
4. Compute its area and exact real transform domain. Explain why the three measurements would not identify a unique response without the three-mode assumption.

Question 9 – Solution

Strategy. The assumed modes turn derivative data into a finite linear system. A negative coefficient need not make their sum negative.

Step 1: Solve and prove uniqueness. The data give

$$c_1 + c_2 + c_3 = 0, \quad -c_1 - 2c_2 - 3c_3 = 0, \quad c_1 + 4c_2 + 9c_3 = 2.$$

Adding the first two equations gives $c_2 = -2c_3$, and the first then gives $c_1 = c_3$. Substitution into the third gives $2c_3 = 2$. Every coefficient is forced, proving uniqueness: $(c_1, c_2, c_3) = (1, -2, 1)$.

Step 2: Assemble the rational transform. The table yields

$$F = \frac{1}{s+1} - \frac{2}{s+2} + \frac{1}{s+3} = \frac{2}{(s+1)(s+2)(s+3)}.$$

Near zero, $f = t^2 + O(t^3)$, so its leading transform is $2/s^3$, matching the quotient. The $1/s$ and $1/s^2$ terms cancel exactly.

Step 3: Establish positivity and the peak. Factor the function as $f = e^{-t}(1 - e^{-t})^2$. It is positive for every $t > 0$ and zero at 0. Put $x = e^{-t} \in (0, 1]$. The function $x(1-x)^2$ has derivative $(1-x)(1-3x)$ with respect to x ; along increasing t its unique interior maximum is at $x = 1/3$. Therefore

$$t_{\max} = \ln 3, \quad f(t_{\max}) = 4/27.$$

Step 4: Check area, domain and identifiability. The area is $1 - 2/2 + 1/3 = 1/3$. The nonzero e^{-t} tail gives exact real domain $s > -1$. Without the mode restriction, adding any multiple of $t^3 e^{-t}$ would preserve all three measured derivatives but change the response. Thus the identification is unique within the stated model, not among all smooth functions.

