

Question 8

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

Consider the transform $F(s) = s/(s+1)$. Allow a full-mass impulse at the origin, with $\mathcal{L}\{\delta(t)\} = 1$ and pre-impact data at 0^- . For ordinary candidates, restrict to functions continuous near zero and of exponential order. The input with transform F drives

$$y' + 2y = f, \quad y(0^-) = 0.$$

Tasks

1. Explain why F cannot be the transform of an ordinary candidate in the stated class. Recover its distributional inverse.
2. Find $Y(s)$ and the complete response for $t \geq 0$, including the jump at zero.
3. Find the positive zero, the global minimum, and the total response area. Explain how a nonzero response can have zero area.
4. Verify the post-impact equation and the equivalent ordinary restart. State the exact real transform domain of the response.

Question 8 – Solution

Strategy. Divide an improper rational expression before consulting ordinary-function rows. Its polynomial part may represent an impulse.

Step 1: Identify the input class. For an ordinary candidate continuous near zero and of exponential order, its Laplace transform tends to zero as $s \rightarrow \infty$: the bounded initial piece has integral $O(1/s)$ and the exponentially bounded tail tends to zero. But $s/(s+1) \rightarrow 1$. Polynomial division gives

$$F = 1 - \frac{1}{s+1}, \quad \boxed{f = \delta(t) - e^{-t}}.$$

The impulse supplies the nonvanishing constant term.

Step 2: Transform the equation and account for the jump. With the stated pre-impact convention, $(s+2)Y = F$, so

$$Y = \frac{s}{(s+1)(s+2)} = -\frac{1}{s+1} + \frac{2}{s+2}, \quad \boxed{y = 2e^{-2t} - e^{-t} \quad (t \geq 0)}.$$

Write $[y]_0 = y(0^+) - y(0^-)$. Integrating across zero gives $[y]_0 = 1$, consistent with $y(0^+) = 1$. The negative exponential input has no impulse mass.

Step 3: Find the sign changes and area. The unique positive zero is $t = \ln 2$. The derivative is $y' = e^{-t} - 4e^{-2t}$, negative before $\ln 4$ and positive afterward. Thus the global minimum is $\boxed{y(\ln 4) = -1/8}$.

The area is $2/2 - 1 = \boxed{0}$: positive and negative parts cancel. For comparison, the signed input area is $1 - 1 = 0$ when the full initial impulse is included.

Step 4: Verify the ordinary restart and convergence. For $t > 0$, direct substitution gives $y' + 2y = -e^{-t}$. The same response solves this ordinary equation with $y(0^+) = 1$. Adding another initial impulse to that restart would count it twice. The tail is asymptotic to $-e^{-t}$, so the exact real transform domain is $\boxed{s > -1}$. The figure shows only the post-impact branch on $t \geq 0$; the filled initial point records the prescribed right-hand value.

