

**Question 7**

Use causal one-sided Laplace transforms. Write  $H_a(t) = 0$  for  $t < a$  and 1 for  $t \geq a$ . Ordinary functions are zero for  $t < 0$ . Justify the table entries and operational rules you use; give exact expressions.

For  $t > 0$ , let

$$g(t) = \frac{e^{-t} - e^{-3t}}{t}.$$

A rational-only table offers no obvious row for this function.

**Tasks**

1. Extend  $g$  continuously to zero and represent it as an integral of exponential functions over a finite parameter interval.
2. Use that representation to derive its transform, carefully justifying interchange and specifying a real domain.
3. Prove positivity and strict decrease on  $[0, \infty)$ , and compute the total area.
4. Find the exact real transform domain, including a proof of divergence at its boundary. Explain why an algebraically real continuation of a logarithm is not sufficient evidence of transform convergence.

## Question 7 – Solution

**Strategy.** Integrating a familiar exponential row over its rate parameter generates a nonrational row.

**Step 1: Remove the time singularity.** Direct integration in  $a$  gives

$$g(t) = \int_1^3 e^{-at} da, \quad g(0) = 2.$$

This formula also proves continuity at zero without dividing by  $t$ .

**Step 2: Integrate the table row.** For real  $s > -1$ , the nonnegative double integral is finite, and interchange gives

$$G(s) = \int_1^3 \int_0^\infty e^{-(s+a)t} dt da = \int_1^3 \frac{da}{s+a} = \ln \frac{s+3}{s+1}.$$

Both logarithm arguments are positive in this domain. Finiteness follows also from  $s+a \geq s+1 > 0$  on the finite parameter interval.

**Step 3: Establish shape and area.** The integrand is positive, so  $g(t) > 0$ . Parameter integration on a bounded interval permits differentiation at every  $t \geq 0$ :

$$g'(t) = - \int_1^3 a e^{-at} da < 0.$$

Thus the extension decreases strictly from 2, approaching zero. Since 0 lies in the transform domain, its area is  $G(0) = \ln 3$ .

**Step 4: Check the boundary from the original integral.** At  $s = -1$ , the transformed integrand is  $(1 - e^{-2t})/t$  for  $t > 0$ . For  $t \geq 1$  it is at least  $(1 - e^{-2})/t$ , whose integral diverges. For  $s < -1$  the positive integrand is even larger on the tail. Therefore  $s > -1$  is the exact real domain. For example, the logarithm of  $(s+3)/(s+1)$  is real again when  $s < -3$ , but that algebraic expression cannot represent the divergent defining integral. The table entry must travel with its domain and the continuous value at zero.