

**Question 6**

Use causal one-sided Laplace transforms. Write  $H_a(t) = 0$  for  $t < a$  and 1 for  $t \geq a$ . Ordinary functions are zero for  $t < 0$ . Justify the table entries and operational rules you use; give exact expressions.

Let  $p$  be the continuous periodic triangular signal of period 2 with

$$p(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ 2-t, & 1 \leq t < 2, \end{cases} \quad p(t+2) = p(t).$$

A standard table lists delayed ramps but does not list this periodic function.

**Tasks**

1. Derive the periodic-transform formula by splitting the half-line into cycles. Apply it to this signal.
2. Write a locally finite series of delayed ramps for  $p$  and use it to obtain the same transform independently.
3. Find the exact real transform domain. Compute  $\lim_{s \rightarrow 0^+} sP(s)$  and interpret it using the mean over one period.
4. Explain why the cycle mean is not a pointwise final value, and determine the first two leading terms of  $P(s)$  as  $s \rightarrow 0^+$ .

## Question 6 – Solution

**Strategy.** A one-cycle transform and a geometric series supply the periodic row; delayed ramps independently encode its slope changes.

**Step 1: Sum the cycles.** For  $s > 0$ , boundedness gives absolute convergence. Splitting into intervals  $[2n, 2n+2]$  and translating yields

$$P(s) = \frac{\int_0^2 e^{-st} p(t) dt}{1 - e^{-2s}} = \frac{(1 - e^{-s})^2}{s^2(1 - e^{-2s})} = \frac{1 - e^{-s}}{s^2(1 + e^{-s})}.$$

The one-cycle integral is  $(1 - e^{-s})^2/s^2$ , obtained by integration by parts on  $[0, 1]$  and  $[1, 2]$ .

**Step 2: Recover the slope changes.** Write  $R_a = (t - a)H_a(t)$ . The series

$$p(t) = \sum_{n=0}^{\infty} [R_{2n} - 2R_{2n+1} + R_{2n+2}]$$

is locally finite; each bracket is a unit triangular pulse on  $[2n, 2n+2]$ . Its transform is  $e^{-2ns}(1 - 2e^{-s} + e^{-2s})/s^2$ . Summing for  $s > 0$  gives the same  $P$ . These brackets, rather than the individual growing ramps across all cycles, also make absolute convergence clear.

**Step 3: Check convergence and the mean.** The signal is bounded, nonnegative and has area 1 per period. Thus the exact real domain is  $s > 0$ : at  $s = 0$  the areas sum to infinity, and for  $s < 0$  the weighted cycle contributions grow. Using  $(1 - e^{-s})/(1 + e^{-s}) = \tanh(s/2)$ ,  $\lim_{s \rightarrow 0^+} sP(s) = \boxed{1/2}$ , the cycle mean  $1/2$ .

**Step 4: Distinguish the limit notions.** The samples  $p(2n) = 0$  and  $p(2n+1) = 1$  rule out a pointwise final value. Taylor expansion of the last quotient gives

$$P(s) = \frac{1}{2s} - \frac{s}{24} + O(s^3).$$

The leading pole records the mean, while the constant term vanishes. The figure displays continuous corners and equal successive cycles.

