

Question 5

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A table contains the infinite-tail entry $te^{-t} \leftrightarrow 1/(s+1)^2$. A finite experiment instead uses

$$w(t) = \begin{cases} te^{-t}, & 0 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

A student subtracts $e^{-2s}/(s+1)^2$ to remove the tail.

Tasks

1. Explain why the proposed subtraction is incorrect, and express the actual tail as a delayed function with its internal clock made explicit.
2. Derive $W(s)$ from table operations and verify it by integration over the finite experiment.
3. Find the exact real transform domain and the removable value at $s = -1$. Explain why the infinite-tail table domain does not restrict this finite-window transform.
4. Find the area, the global maximum and the one-sided values at the cutoff. Draw the finite-window response accurately on the solution page.

Question 5 – Solution

Strategy. Removing an old tail requires its original age, not a fresh copy starting at zero.

Step 1: Identify the tail that must be removed. For $t = 2 + u$, the old signal is $e^{-2}(u + 2)e^{-u}$. Thus

$$w = te^{-t} - e^{-2}H_2(t)[(t - 2) + 2]e^{-(t-2)}.$$

The proposed subtraction instead removes $H_2(t)(t - 2)e^{-(t-2)}$, which has both the wrong age polynomial and the wrong exponential scale.

Step 2: Transform and check directly. For $s \neq -1$, the correct expression is

$$W(s) = \frac{1 - e^{-2(s+1)}[1 + 2(s + 1)]}{(s + 1)^2}.$$

Initially table operations justify it for $s > -1$. Independently put $p = s + 1$: $W = \int_0^2 te^{-pt} dt$. Integration by parts gives $[1 - e^{-2p}(1 + 2p)]/p^2$, agreeing with the table calculation and extending its formula to $p \neq 0$ of either sign.

Step 3: Remove the singularity and identify the domain. A bounded finite interval makes this transform finite for every real s . At $s = -1$ the defining integral is $\int_0^2 t dt = [2]$. The numerator's Taylor expansion is $2p^2 + O(p^3)$, so the apparent pole is removable. The two infinite-tail terms need only be transformed separately on their common domain; their difference has a larger actual domain.

Step 4: Record the geometric checks. The area is $W(0) = [1 - 3e^{-2}]$. Since $(te^{-t})' = (1 - t)e^{-t}$, the unique global maximum is $1/e$ at $t = 1$. At the cutoff, $w(2^-) = 2e^{-2}$ while $w(2) = w(2^+) = 0$. The open dot marks the excluded left-hand value and the filled dot the actual value at 2; no slanted segment bridges the jump.

