

Question 4

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

An unknown function has transform

$$F(s) = \frac{As + B}{s^2 + 4}, \quad s > 0,$$

where A, B are real. Measurements give $f(0) = 1$ and $f'(0) = -2$.

Tasks

1. Invert the general transform by matching numerators to table entries, then recover A, B from the data.
2. For the recovered function, find its amplitude, all positive zeros, and the conserved quantity $f'(t)^2 + 4f(t)^2$.
3. Classify all real pairs (A, B) for which the inverse is nonnegative for every $t \geq 0$. Prove the classification.
4. A student argues that $\lim_{s \rightarrow 0^+} sF(s) = 0$ proves $\lim_{t \rightarrow \infty} f(t) = 0$. Refute the conclusion with two sequences of times and identify the missing hypothesis in that use of the final-value theorem.

Question 4 – Solution

Strategy. Matching a denominator is insufficient: the numerator determines the phase, initial state and sign changes.

Step 1: Match and use the data. The table gives $f = A \cos 2t + (B/2) \sin 2t$, so $f(0) = A$ and $f'(0) = B$. Therefore $A = 1$, $B = -2$ and

$$f(t) = \cos 2t - \sin 2t = \sqrt{2} \cos(2t + \pi/4).$$

Step 2: Determine the motion. The amplitude is $\sqrt{2}$. The positive zeros are $t = \pi/8 + k\pi/2$, $k = 0, 1, 2, \dots$. Direct differentiation gives $f'' + 4f = 0$, so $(f'^2 + 4f^2)' = 2f'(f'' + 4f) = 0$. Its initial value is $(-2)^2 + 4(1)^2 = \boxed{8}$.

Step 3: Prove the positivity classification. If $(A, B) \neq (0, 0)$, the amplitude $\sqrt{A^2 + B^2/4}$ is positive. The sinusoid attains both that amplitude and its negative at arbitrarily large nonnegative times. It cannot be nonnegative on the whole half-line. Only $\boxed{A = B = 0}$ works. An alternative proof uses zero integral over one period: a continuous nonnegative function with zero integral must vanish.

Step 4: Check the claimed final value. Although $sF(s) \rightarrow 0$ as $s \rightarrow 0^+$, the selected solution obeys $f(n\pi) = 1$ and $f(n\pi + \pi/2) = -1$. Hence it has no time limit. The usual rational final-value theorem requires the poles of $sF(s)$ to lie strictly in the left half-plane after cancellation. Here the uncanceled poles are $s = \pm 2i$, supporting persistent oscillations. An algebraic limit of the transform does not by itself establish a limit of the function.