

Question 3

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A short table gives only $\mathcal{L}\{\sin(at)\} = a/(s^2 + a^2)$, where $a > 0$. You need inverses of

$$A_a(s) = \frac{s}{(s^2 + a^2)^2}, \quad B_a(s) = \frac{1}{(s^2 + a^2)^2}.$$

Tasks

1. Derive the inverse of A_a by differentiating a transform with respect to s .
2. Derive the inverse of B_a by differentiating the given sine entry with respect to a . Explain why the parameter differentiation is justified for real $s > 0$.
3. Check the first nonzero Taylor term at $t = 0$ against the leading large- s term of each transform.
4. Find both pointwise limits as $a \rightarrow 0^+$ and explain why substituting $a = 0$ into an unsimplified inverse formula is invalid.

Question 3 – Solution

Strategy. Generate missing table rows through operations whose normalization can be checked independently.

Step 1: Differentiate with respect to s . Since $-\frac{d}{ds}[a/(s^2 + a^2)] = 2as/(s^2 + a^2)^2$,

$$\mathcal{L}^{-1}\{A_a\} = \frac{t \sin(at)}{2a}.$$

The factor $2a$ must be divided out.

Step 2: Differentiate with respect to a . The parameter derivative gives $\mathcal{L}\{t \cos(at)\} = (s^2 - a^2)/(s^2 + a^2)^2$. For fixed $s > 0$, te^{-st} dominates the differentiated integrand uniformly in a , justifying differentiation under the integral. Subtracting this row from $\mathcal{L}\{\sin(at)/a\}$ yields

$$\frac{1}{s^2 + a^2} - \frac{s^2 - a^2}{(s^2 + a^2)^2} = \frac{2a^2}{(s^2 + a^2)^2}.$$

Thus

$$\mathcal{L}^{-1}\{B_a\} = \frac{\sin(at) - at \cos(at)}{2a^3}.$$

Step 3: Check the onset powers. The inverses start as $t^2/2 + O(t^4)$ and $t^3/6 + O(t^5)$. Their leading transforms are respectively $1/s^3$ and $1/s^4$, matching A_a and B_a . This checks both factorials and the cancellation of the linear term in the second numerator.

Step 4: Remove the apparent parameter singularities. At each fixed t , the Taylor expansions give limits $t^2/2$ and $t^3/6$. Their transforms are $1/s^3$ and $1/s^4$, agreeing with the limits of the rational functions for $s > 0$. Directly inserting $a = 0$ into the inverse quotients produces $0/0$; the limits require cancellation first. The figure uses $a = 1$ and shows that the repeated denominator can produce oscillations with increasing size, unlike a single sine entry.

