

Question 2

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

For $a > 0$, define the delay and time-multiplication operations

$$(D_a f)(t) = H_a(t)f(t-a), \quad (Mf)(t) = tf(t).$$

Take $f(t) = e^{-t}$, and compare $p = D_a(Mf)$ with $q = M(D_a f)$.

Tasks

1. Derive the transform rules for D_a and M from their defining integrals. State a domain where the derivations are justified for this f .
2. Find p, q and their transforms. Explain exactly why applying the two operations in opposite orders changes the answer.
3. Prove the general identity $MD_a f - D_a Mf = aD_a f$ whenever the expressions are defined, and verify its transform.
4. For $a = 2$, invert $e^{-2s}/(s+1)^2$ and $-\frac{d}{ds}[e^{-2s}/(s+1)]$. Compute the area of each inverse and use it to distinguish them.

Question 2 – Solution

Strategy. A delayed clock measures $t - a$; multiplication after the delay still uses the original clock t .

Step 1: Derive the rules. The substitution $u = t - a$ gives $\mathcal{L}\{D_a f\} = e^{-as}F(s)$. Differentiating the transform integral with respect to s gives $\mathcal{L}\{Mf\} = -F'(s)$. For $f = e^{-t}$ and its delayed polynomial multiples, $s > -1$ makes these integrals absolutely convergent and permits this differentiation, locally dominated by an integrable exponential times a polynomial.

Step 2: Keep the clocks explicit. Here

$$p = H_a(t)(t - a)e^{-(t-a)}, \quad q = H_a(t)te^{-(t-a)},$$

so

$$P = \frac{e^{-as}}{(s+1)^2}, \quad Q = -\frac{d}{ds} \frac{e^{-as}}{s+1} = e^{-as} \left[\frac{a}{s+1} + \frac{1}{(s+1)^2} \right].$$

The extra term comes from differentiating the delay factor itself.

Step 3: Prove the operation identity. For $t < a$ both sides vanish. For $t \geq a$, subtraction gives $tf(t-a) - (t-a)f(t-a) = af(t-a)$. Therefore

$$\boxed{MD_a f - D_a Mf = aD_a f.}$$

In transform language, $-(e^{-as}F)' - e^{-as}(-F') = ae^{-as}F$, the same identity by the product rule.

Step 4: Invert and distinguish the two expressions. For $a = 2$ the first inverse is $H_2(t)(t-2)e^{-(t-2)}$, while the second is $H_2(t)te^{-(t-2)}$. Their areas, after putting $u = t - 2$, are

$$\int_0^\infty ue^{-u} du = \boxed{1}, \quad \int_0^\infty (u+2)e^{-u} du = \boxed{3}.$$

Their transforms at zero have these same values. The first starts continuously from zero at the delay; the second has right-hand value 2 there. Isolated endpoint conventions do not change either area.