

Question 9

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Let f have period two, with $f(t) = 1$ for $0 \leq t < 1$ and $f(t) = 0$ for $1 \leq t < 2$, repeated for all $t \geq 0$. Use the definition rather than a precomputed periodic-transform formula.

Tasks

1. Split the integral into complete periods, sum the resulting series, and determine the exact real convergence set.
2. Simplify the transform on its convergence interval.
3. Compute $\lim_{s \rightarrow 0^+} sF(s)$ and $\lim_{s \rightarrow \infty} sF(s)$. Relate these to the period average and to the signal just after zero.
4. If the defining integral is truncated after N complete periods, find its exact relative error for $s > 0$. At $s = 0.2$, find the smallest integer N making that error at most 10^{-3} .

Question 9 – Solution

Strategy. A time translation by one period multiplies the weighted area by a fixed factor, producing a geometric series.

Step 1: Build the series and its domain. For $s > 0$, the n th nonzero interval contributes

$$\int_{2n}^{2n+1} e^{-st} dt = e^{-2ns} \frac{1 - e^{-s}}{s}.$$

The ratio e^{-2s} lies in $(0, 1)$, so the sum converges. At $s = 0$, each interval contributes one and the sum diverges. For $s < 0$, $e^{-st} \geq 1$ on each such interval, so it also diverges. The real convergence set is exactly $(0, \infty)$, with absolute convergence since $f \geq 0$.

Step 2: Simplify the geometric sum. Summing gives

$$F(s) = \frac{1 - e^{-s}}{s(1 - e^{-2s})} = \boxed{\frac{1}{s(1 + e^{-s})}, \quad s > 0.}$$

The integral over only one period is the first term of this sum, not the transform of the endlessly repeated signal.

Step 3: Interpret the two normalized limits. The formula gives

$$\boxed{\lim_{s \rightarrow 0^+} sF(s) = \frac{1}{2}, \quad \lim_{s \rightarrow \infty} sF(s) = 1.}$$

The first value is the period average $(1/2) \int_0^2 f(t) dt = 1/2$. The second agrees with $f(0^+) = 1$: large s weights short times most strongly. These normalized limits do not assert that $F(0)$ exists; it does not.

Step 4: Quantify finite-window error. Writing $F_N = \int_0^{2N} e^{-st} f(t) dt$, the finite geometric sum gives

$$F_N = F(s)(1 - e^{-2Ns}), \quad \boxed{\frac{F(s) - F_N}{F(s)} = e^{-2Ns}.}$$

At $s = 0.2$, the requirement is $e^{-0.4N} \leq 10^{-3}$, so $N \geq \ln(1000)/0.4 \approx 17.27$. The least integer is $\boxed{N = 18}$, corresponding to a cutoff time of 36. Endpoint assignments do not affect the integrals; the graph uses the values specified in the question.

