

Question 8

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

An unknown signal belongs to the family $f(t) = 0$ for $0 \leq t < a$ and $f(t) = A$ for $t \geq a$, where $A > 0$ and $a \geq 0$. Exact measurements are

$$F(1) = 2/e, \quad F(2) = e^{-2}.$$

Tasks

1. Derive the transform and its real convergence set directly, then recover A and a from the measurements.
2. For arbitrary positive measurements $M_1 = F(1)$ and $M_2 = F(2)$, give a necessary and sufficient consistency condition for this family and explicit recovery formulas.
3. For $h(t) = e^{-t}(t^2 - 5t/3 + 1/3)$, compute its transform at $s > 0$ directly by integration by parts and show that it vanishes at both measured parameters.
4. Use h to explain precisely why the two measurements identify the signal within the stated family but do not identify an arbitrary real-valued signal.

Question 8 – Solution

Strategy. Distinguish uniqueness of a few model parameters from uniqueness among all possible functions.

Step 1: Integrate and calibrate. For $s > 0$,

$$F(s) = A \int_a^\infty e^{-st} dt = \frac{Ae^{-as}}{s}.$$

For $s \leq 0$, the positive tail integral diverges. Dividing the two measured equations gives $2F(2)/F(1) = e^{-a} = e^{-1}$, so $\boxed{a = 1, A = 2}$. These values reproduce both measurements.

Step 2: Characterize all consistent data. For positive M_1, M_2 , the family requires $2M_2/M_1 = e^{-a} \in (0, 1]$. Thus consistency is exactly $\boxed{2M_2 \leq M_1}$. In that case the unique parameters are

$$\boxed{a = \ln \frac{M_1}{2M_2}, \quad A = \frac{M_1^2}{2M_2}.$$

They satisfy $a \geq 0, A > 0$ and both equations, proving sufficiency as well as necessity. Equality corresponds to onset at zero.

Step 3: Construct an invisible perturbation. Set $b = s + 1 > 0$. Integration by parts gives the three integrals of $1, t, t^2$ against e^{-bt} as $1/b, 1/b^2, 2/b^3$. Therefore

$$H(s) = \frac{2}{(s+1)^3} - \frac{5}{3(s+1)^2} + \frac{1}{3(s+1)} = \boxed{\frac{(s-1)(s-2)}{3(s+1)^3}}.$$

The exponential boundary terms vanish, justifying those integrations. In particular, $H(1) = H(2) = 0$, although h is not the zero function.

Step 4: State the identification limit. For every real λ , the signal $f + \lambda h$ has a convergent transform for $s > 0$. Splitting its defining integral gives $F(s) + \lambda H(s)$, so both recorded measurements are unchanged. Distinct λ give distinct real-valued signals; for nonzero λ they are not members of the original two-parameter family. The recovery formulas establish uniqueness only under that family assumption, not among arbitrary signals.