

Question 7

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

For each integer $n \geq 1$, let $w_n = e^{-2n^2}$. Define a nonnegative signal by

$$f(t) = \begin{cases} e^{n^2}, & n \leq t \leq n + w_n \text{ for some } n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

A function is of exponential order if $|f(t)| \leq M e^{at}$ for all sufficiently large t , for some constants $M > 0$ and real a .

Tasks

1. Explain why f is piecewise continuous on every finite interval but is not of exponential order.
2. Prove that its Laplace integral nevertheless converges absolutely for every real s .
3. Express $F(s)$ as a convergent series, treating $s = 0$ separately.
4. Explain how the spike widths resolve the apparent conflict between arbitrarily large heights and convergence, and why the usual exponential-order theorem is not contradicted.

Question 7 – Solution

Strategy. The integral depends on weighted areas, not just pointwise heights. Quantify the widths before invoking a growth criterion.

Step 1: Examine local regularity and height. Since $w_n < 1$, the spike intervals are disjoint. Any bounded time interval meets only finitely many spikes, each with finite height and finite one-sided limits. Thus f is piecewise continuous locally. If an exponential-order bound held, evaluating at $t = n$ for large integers would give $e^{n^2} \leq Me^{an}$, or $n^2 - an \leq \ln M$. The left side tends to infinity, a contradiction.

Step 2: Estimate each weighted area. Fix real s . On the n th spike, $t \leq n + 1$ and $e^{-st} \leq e^{|s|(n+1)}$. Its contribution is therefore at most

$$e^{n^2} w_n e^{|s|(n+1)} = e^{-n^2 + |s|(n+1)}.$$

For sufficiently large n , $|s|(n+1) \leq n^2/2$, so these terms are at most $e^{-n^2/2} \leq e^{-n/2}$. The comparison series converges. Since the integrand is nonnegative, the bound proves absolute convergence for every real s .

Step 3: Sum the exact contributions. Split finite integrals into their finitely many spike intervals, then pass to the limit using the convergent positive series. For $s \neq 0$,

$$F(s) = \sum_{n=1}^{\infty} e^{n^2 - sn} \frac{1 - e^{-sw_n}}{s}.$$

The quotient is positive even for negative s . At $s = 0$, each contribution is simply height times width, giving

$$F(0) = \sum_{n=1}^{\infty} e^{-n^2}.$$

No closed form for this convergent series is required.

Step 4: Interpret the theorem correctly. The heights grow faster than every exponential, but the widths shrink fast enough that their areas are e^{-n^2} . Any fixed exponential weight still leaves a summable sequence of areas. The common existence theorem provides sufficient conditions; exponential order is not necessary for an individual function to have a Laplace transform. This example establishes that distinction by direct estimates, rather than by trying to apply a theorem whose hypothesis fails.