

Question 6

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Compare $f_+(t) = e^{t^2}$ and $f_-(t) = e^{-t^2}$ for $t \geq 0$, and write their transforms as F_+ and F_- .

Tasks

1. Determine the real convergence set of each transform directly from the defining integral.
2. Complete the square to express $F_-(s)$ through a Gaussian tail for every real s . No special-function notation is required.
3. For a truncation at T with $T + s/2 > 0$, prove an explicit upper bound for the omitted tail of $F_-(s)$.
4. At $s = -2$, find the smallest integer $T > 1$ for which your bound certifies an error at most 10^{-4} . Explain why negative s does not destroy convergence for f_- .

Question 6 – Solution

Strategy. Compare the quadratic growth or decay with the linear term supplied by the Laplace weight, and use the completed square to bound numerical truncation.

Step 1: Classify both integrals. For any real s , $t^2 - st \geq 0$ when $t \geq \max\{s, 0\}$, so $e^{t^2-st} \geq 1$ on a half-line. Hence F_+ diverges for every real s . For F_- ,

$$-t^2 - st = -(t + s/2)^2 + s^2/4.$$

A shifted Gaussian has an integrable tail and no finite-endpoint singularity, so F_- converges absolutely for every real s .

Step 2: Express the exact transform. Substitution $u = t + s/2$ gives

$$F_-(s) = e^{s^2/4} \int_{s/2}^{\infty} e^{-u^2} du, \quad s \in \mathbb{R}.$$

The lower endpoint is allowed to be negative; the finite portion of the integral then remains harmless. This definite integral is an exact answer.

Step 3: Bound the omitted tail. Let $a = T + s/2 > 0$. Since $1 \leq u/a$ for $u \geq a$,

$$\int_a^{\infty} e^{-u^2} du \leq \frac{1}{a} \int_a^{\infty} u e^{-u^2} du = \frac{e^{-a^2}}{2a}.$$

Consequently the positive truncation error satisfies

$$\int_T^{\infty} e^{-t^2-st} dt \leq \frac{e^{-T^2-sT}}{2T+s}.$$

The condition $2T + s > 0$ is part of the bound and cannot be dropped.

Step 4: Certify a cutoff. For $s = -2$, the bound is $e^{-T^2+2T}/(2T-2)$, decreasing for $T > 1$. At $T = 3$ it is $e^{-3}/4 \approx 0.01245$, while at $T = 4$ it is $e^{-8}/6 \approx 5.59 \times 10^{-5}$. Thus the smallest integer cutoff certified by this bound is $\boxed{T = 4}$. Negative s makes the weight grow exponentially, but the quadratic decay still dominates. The graph shows the weighted integrands, whose areas give the transforms.

