

Question 5

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Let $f(t) = \sin t/t$ for $t > 0$, with $f(0) = 1$. In this problem examine parameters $s \geq 0$. You may derive the positive-parameter transform using $\sin t/t = \int_0^1 \cos(ut) du$.

Tasks

1. For $s > 0$, justify interchanging the two integrations and evaluate $F(s)$.
2. For $s \geq 0$ and $R > 0$, prove the tail bound $|\int_R^\infty e^{-st} \sin t/t dt| \leq 2/R$, including existence of the tail at $s = 0$.
3. Use that uniform tail bound and finite-interval convergence as $s \rightarrow 0^+$ to determine $F(0)$.
4. Determine whether the integral at $s = 0$ converges absolutely, and contrast it with the positive-parameter integrals.

Question 5 – Solution

Strategy. A uniform tail estimate can justify passage to the boundary parameter; absolute and conditional convergence must still be distinguished.

Step 1: Evaluate for positive damping. For $s > 0$, the double integral of the absolute integrand is at most $\int_0^\infty e^{-st} dt = 1/s$, so interchange is justified. Integration by parts gives $\int_0^\infty e^{-st} \cos(ut) dt = s/(s^2 + u^2)$. Therefore

$$F(s) = \int_0^1 \frac{s}{s^2 + u^2} du = \boxed{\arctan(1/s), \quad s > 0.}$$

Step 2: Control the tail uniformly. Set $w(t) = e^{-st}/t$ for $t \geq R$. For every $s \geq 0$, w decreases to zero and $w' \leq 0$. Integration by parts on $[R, L]$ gives

$$\int_R^L w(t) \sin t dt = [-w(t) \cos t]_R^L + \int_R^L w'(t) \cos t dt.$$

The integral of $|w'|$ is $w(R) - w(L)$, so the right side has magnitude at most $2w(R) \leq 2/R$. Applying the same bound with any larger lower endpoint proves the Cauchy criterion as $L \rightarrow \infty$, including at $s = 0$. The bound persists for the improper tail.

Step 3: Pass to the boundary parameter. On each fixed $[0, R]$, $e^{-st}f(t) \rightarrow f(t)$ uniformly as $s \rightarrow 0^+$, since f is continuous and bounded there. The tails at both s and zero have magnitude at most $2/R$. Thus

$$|F(s) - F(0)| \leq \left| \int_0^R (e^{-st} - 1)f(t) dt \right| + 4/R.$$

First choose large R , then let $s \rightarrow 0^+$. This proves continuity at the boundary and hence $\boxed{F(0) = \pi/2}$.

Step 4: Test absolute convergence. On $[n\pi + \pi/6, n\pi + 5\pi/6]$, $|\sin t| \geq 1/2$ and $t \leq (n+1)\pi$. Each such interval contributes at least $1/[3(n+1)]$ to $\int |\sin t|/t dt$. The harmonic series diverges, so convergence at zero is conditional. For $s > 0$, $|\sin t/t| \leq 1$ gives absolute convergence by comparison with e^{-st} . The boundary value here exists as an ordinary improper integral, not merely as a limit of damped values.