

Question 4

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

For $t > 0$, let $f(t) = t^{-1/2}e^{-t}$; assign any finite value to $f(0)$. For the defining integral, treat zero as an improper endpoint as well as infinity. You may use $\int_0^\infty e^{-u^2} du = \sqrt{\pi}/2$.

Tasks

1. Analyze convergence near zero and near infinity separately, and find the exact real convergence set.
2. Evaluate $F(s)$ on that set by a substitution.
3. Repeat the endpoint analysis for $h(t) = t^{-1}e^{-t}$, $t > 0$. Does exponential decay guarantee a transform for this function?
4. Explain why the first example does not contradict a theorem giving piecewise continuity on each finite interval and exponential order as sufficient conditions for existence.

Question 4 – Solution

Strategy. An improper transform can fail at either endpoint. Rapid decay at infinity cannot repair a nonintegrable singularity at zero.

Step 1: Separate the endpoints. Write $a = s + 1$. Near zero, e^{-at} is bounded above and below by positive constants, and $\int_0^1 t^{-1/2} dt = 2$, so this endpoint is integrable for every fixed real s . At infinity, $a > 0$ gives exponential decay. If $a = 0$, the integral of $t^{-1/2}$ diverges; if $a < 0$, the integrand eventually exceeds one. Hence convergence occurs exactly for $s > -1$, and is absolute.

Step 2: Evaluate the convergent integral. For $a > 0$, use $u = \sqrt{at}$, so $t = u^2/a$ and $dt = 2u du/a$. Then

$$F(s) = \frac{2}{\sqrt{a}} \int_0^\infty e^{-u^2} du = \boxed{\frac{\sqrt{\pi}}{\sqrt{s+1}}, \quad s > -1.}$$

Both the substitution and the square root require the already established condition $a > 0$. An assigned value at the single point zero does not change the improper integral.

Step 3: Test the stronger singularity. For h , the integrand is $t^{-1}e^{-(s+1)t}$. On a sufficiently small interval $(0, \delta)$, its exponential factor is at least $1/2$. Thus

$$\int_0^\delta t^{-1} e^{-(s+1)t} dt \geq \frac{1}{2} \int_0^\delta \frac{dt}{t} = \infty.$$

This holds for every real s . Consequently h has no transform under this definition, even for parameters that give fast decay at infinity.

Step 4: Interpret the sufficient theorem. The first function is unbounded as $t \rightarrow 0^+$, so it does not satisfy the usual finite-limit requirement for piecewise continuity on $[0, A]$. Yet its endpoint singularity is integrable, and the direct calculation establishes existence for $s > -1$. A sufficient theorem guarantees existence when its hypotheses hold; failure of a hypothesis does not prove nonexistence. Here direct endpoint analysis gives the sharper conclusion.