

Question 3

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Let $f(t) = \sin 3t$ for $t \geq 0$. Its truncated transform is $F_R(s) = \int_0^R e^{-st} \sin 3t dt$.

Tasks

1. Evaluate $F_R(s)$ for real s , retaining its upper-endpoint contribution.
2. Find the exact real convergence interval for $F(s)$ and determine whether convergence there is absolute.
3. At $s = 0$, exhibit two sequences of upper endpoints giving different values of $F_R(0)$.
4. Compare $\lim_{s \rightarrow 0^+} F(s)$ with the existence of $F(0)$. Why does a finite parameter limit not justify interchanging the two limits $R \rightarrow \infty$ and $s \rightarrow 0^+$?

Question 3 – Solution

Strategy. Damping by e^{-st} can create convergence that disappears at the boundary parameter.

Step 1: Keep the endpoint term. Two integrations by parts, or differentiation of the resulting antiderivative, give

$$F_R(s) = \frac{3 - e^{-sR}[s \sin 3R + 3 \cos 3R]}{s^2 + 9}.$$

There is no real zero of the denominator, but that alone says nothing about the limit as $R \rightarrow \infty$.

Step 2: Classify convergence. If $s > 0$, the endpoint term vanishes and

$$F(s) = \frac{3}{s^2 + 9}, \quad s > 0.$$

Absolute convergence follows from $|e^{-st} \sin 3t| \leq e^{-st}$. If $s < 0$, take $R_n = 2n\pi/3$. Then $F_{R_n}(s) = 3(1 - e^{|s|R_n})/(s^2 + 9) \rightarrow -\infty$, so the improper integral cannot converge.

Step 3: Test the boundary directly. At $s = 0$, $F_R(0) = (1 - \cos 3R)/3$. For

$$R_n = 2n\pi/3, \quad F_{R_n}(0) = 0; \quad Q_n = (2n+1)\pi/3, \quad F_{Q_n}(0) = 2/3.$$

The two sequences tend to infinity but have different integral values. Thus $F(0)$ does not exist, and the full real convergence interval is $(0, \infty)$.

Step 4: Separate the two limiting procedures. The damped transforms satisfy $\lim_{s \rightarrow 0^+} F(s) = 1/3$. In contrast, setting $s = 0$ first produces an oscillating truncated integral with no limit. For fixed R , the limit in s exists, but convergence in R is not uniform as damping disappears. The value $1/3$ is a limit of damped integrals, not the value of the undamped improper integral. The graph compares accumulated integrals, not the original sine signal.

