

Question 1

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Let $f(t) = e^{2t}(1+t)$ for $t \geq 0$. A student obtains the expression $1/(s-2) + 1/(s-2)^2$ and claims that it gives a transform value at every real $s \neq 2$.

Tasks

1. Evaluate the truncated defining integral from zero to R , treating $s = 2$ separately.
2. Determine exactly which real s give convergence, and find $F(s)$ on that set.
3. Evaluate the student's expression at $s = 1$ and explain why it cannot be the Laplace integral there.
4. Prove directly from the integral, without differentiating the final formula, that F is positive and strictly decreasing throughout its convergence interval.

Question 1 – Solution

Strategy. Keep the upper endpoint finite until the parameter-dependent boundary terms have been examined.

Step 1: Evaluate the finite integral. Write $a = s - 2$. Integration by parts gives, for $a \neq 0$,

$$I_R = \int_0^R (1+t)e^{-at} dt = \frac{1}{a} + \frac{1}{a^2} - e^{-aR} \left[\frac{1+R}{a} + \frac{1}{a^2} \right].$$

At $a = 0$, direct integration gives $I_R = R + R^2/2$. The formula with denominators cannot be used at that parameter.

Step 2: Determine the convergence set. For $a > 0$, both e^{-aR} and Re^{-aR} tend to zero. Therefore

$$F(s) = \frac{1}{s-2} + \frac{1}{(s-2)^2}, \quad s > 2.$$

For $a = 0$, the displayed polynomial diverges. For $a < 0$, the integrand $(1+t)e^{|a|t}$ is at least one, so $I_R \geq R \rightarrow \infty$. Thus $s > 2$ is the entire real convergence set, and convergence there is absolute.

Step 3: Reject an algebraic continuation as an integral value. At $s = 1$, the rational expression equals $-1 + 1 = 0$. But the actual integral is

$$I_R = \int_0^R (1+t)e^t dt = Re^R \rightarrow \infty.$$

The discarded boundary term grows instead of vanishing. A finite value of the algebraic expression outside the convergence interval does not assign a value to the defining improper integral.

Step 4: Prove order properties from the kernel. For $s > 2$, the integrand is positive, so $F(s) > 0$. If $s_2 > s_1 > 2$, then for every $t > 0$,

$$0 < e^{-s_2 t} f(t) < e^{-s_1 t} f(t).$$

Both integrals converge, and their difference is strictly positive on any fixed interval of positive length inside $(0, \infty)$. Hence $F(s_2) < F(s_1)$. This argument shows why positivity and monotonicity hold, independently of the rational simplification.