

Question 10

For a real parameter c , consider on $\mathbb{R}$

$$y'' - y = e^{-t} + ce^{-2t}, \quad y(0) = y'(0) = 0.$$

Both forcing components tend to zero as $t \rightarrow \infty$.

Tasks

1. Choose resonant and nonresonant trials as appropriate and find the full zero-data solution in terms of c .
2. Find the unique value of c making this solution bounded on $[0, \infty)$.
3. For that tuned value, prove the sign of the solution for every $t > 0$ and find its limiting value. You may use the strict inequality $e^{-t} > 1 - t$ for $t > 0$.
4. If the tuned parameter is replaced by $c + \delta$ with $\delta \neq 0$, determine the leading long-time term and its sign. Explain why decaying forcing alone does not guarantee a decaying response.

Question 10 – Solution

Strategy. Determine which forcing combination cancels the growing homogeneous mode after the initial data are fitted.

Step 1: Match and fit. The root -1 is simple, while -2 is not a root. Use $y_p = ate^{-t} + be^{-2t}$. Since $(D^2 - 1)(te^{-t}) = -2e^{-t}$ and $(D^2 - 1)e^{-2t} = 3e^{-2t}$, we obtain $a = -1/2$, $b = c/3$. Fitting the zero data gives

$$y = \left(\frac{1}{4} + \frac{c}{6}\right)e^t + \left(-\frac{1}{4} - \frac{c}{2}\right)e^{-t} - \frac{1}{2}te^{-t} + \frac{c}{3}e^{-2t}.$$

The value is zero because the constant coefficients sum to zero; the slope is zero because the derivative contributions sum to zero.

Step 2: Cancel the unstable mode. Every term except the e^t term tends to zero. Thus boundedness holds exactly when $1/4 + c/6 = 0$, giving $c = -3/2$. The tuned response is

$$y_* = \frac{1}{2}e^{-t}(1 - t - e^{-t}).$$

Step 3: Prove sign and decay. For $t > 0$, the supplied strict exponential inequality gives $1 - t - e^{-t} < 0$, so $y_* < 0$. All its terms decay, hence $y_* \rightarrow 0$ from below. It has zero initial value and slope despite remaining strictly negative afterward.

Step 4: Examine imperfect cancellation. Replacing the tuned value by $-3/2 + \delta$ makes the growing-mode coefficient $\delta/6$. More explicitly, the change in solution is

$$\delta[e^t/6 - e^{-t}/2 + e^{-2t}/3].$$

Consequently $y(t)/e^t \rightarrow \delta/6$: the response tends to positive infinity for $\delta > 0$ and negative infinity for $\delta < 0$. The forcing still decays; the initial-data adjustment excites a growing homogeneous component unless the cancellation is exact.

