

Question 8

Consider the standard finite polynomial–exponential–trigonometric version of undetermined coefficients for constant-coefficient equations. Then examine $y'' + y = 1/(1+t)$ on $I = (-1, \infty)$.

Tasks

1. Which forcings fit the standard finite trial families: $e^{-t}(t^2 + 1)$, $t \sin 3t$, $\cos^2 t$, e^{t^2} , and $1/(1+t)$? Explain any algebraic rewriting needed.
2. Test the rational trial $y_p = A/(1+t)$ for the stated equation and prove that no constant A works.
3. Prove that no finite sum $\sum_{k=1}^N a_k(1+t)^{-k}$ with real coefficients can be a particular solution.
4. Does the failure of these trials imply that an initial-value solution does not exist? State the existence and uniqueness conclusion for any initial point in I .

Question 8 – Solution

Strategy. The standard method depends on closure in a finite differentiation family; a plausible-looking trial outside that family need not close.

Step 1: Classify the forcings. The first two belong to the standard families. The third does too after $\cos^2 t = (1 + \cos 2t)/2$. The functions e^{t^2} and $1/(1+t)$ do not: successive derivatives introduce polynomial factors of ever-increasing degree in the first case and poles of ever-increasing order in the second. The usual finite constant-coefficient matching prescription does not apply to them.

Step 2: Test one inverse power. Put $x = 1 + t > 0$. For $y_p = A/x$,

$$y_p'' + y_p = A/x + 2A/x^3.$$

Equality to $1/x$ would require $A(x^2 + 2) = x^2$ for all $x > 0$. Matching x^2 forces $A = 1$, while matching the constant term forces $2A = 0$, a contradiction.

Step 3: Exclude any finite inverse-power sum. If such a sum were nonzero, choose its largest index N with $a_N \neq 0$. Its second derivative contains

$$a_N N(N+1)x^{-N-2}.$$

No other differentiated term, undifferentiated term or forcing term has that most singular power. Equivalently, multiply the proposed equation by x^{N+2} and let $x \downarrow 0$: the left side tends to $a_N N(N+1) \neq 0$, while the right side tends to zero. The zero sum also fails to give the nonzero forcing.

Step 4: Separate method failure from nonexistence. On I , the normalized coefficients and the forcing $1/(1+t)$ are continuous. For every $t_0 \in I$ and every finite value and slope at t_0 , the regular linear theorem gives exactly one solution on I . A failed finite ansatz is a limitation of the trial class, not a proof that the differential equation lacks solutions.