

Question 7

For $\omega > 0$, let y_ω solve

$$y'' + y = \cos(\omega t), \quad y(0) = y'(0) = 0.$$

Tasks

1. Use undetermined coefficients to find y_ω for $\omega \neq 1$, including the homogeneous correction.
2. At $\omega = 1$, choose a resonant trial and solve the same initial-value problem directly.
3. For every fixed t , compute $\lim_{\omega \rightarrow 1} y_\omega(t)$. Explain why the divergent particular coefficient for $\omega \neq 1$ does not imply a divergent fixed-time limit of the full response.
4. Compare boundedness on $[0, \infty)$ in the resonant and nonresonant cases. Can the convergence as $\omega \rightarrow 1$ be uniform on that entire half-line?

Question 7 – Solution

Strategy. Fit the initial data before taking the frequency limit; the homogeneous correction cancels the divergent constant part.

Step 1: Solve off resonance. The trial $A \cos \omega t$ gives $A(1 - \omega^2) = 1$. Fitting zero data adds $-\cos t/(1 - \omega^2)$, so

$$y_\omega(t) = \frac{\cos(\omega t) - \cos t}{1 - \omega^2}, \quad \omega \neq 1.$$

Both the value and slope at zero vanish.

Step 2: Solve at resonance. Use $t(A \cos t + B \sin t)$. Applying $D^2 + 1$ gives $-2A \sin t + 2B \cos t$, hence $A = 0$, $B = 1/2$. Its data are already zero, so $y_1 = (t/2) \sin t$.

Step 3: Take the fixed-time limit. Differentiating numerator and denominator with respect to ω gives

$$\lim_{\omega \rightarrow 1} y_\omega(t) = \frac{-t \sin t}{-2} = \frac{1}{2} t \sin t.$$

This includes $t = 0$. The particular coefficient and the cosine correction separately diverge, but their full numerator vanishes at the same rate as the denominator. They must be combined before taking the limit.

Step 4: Compare long-time behavior. For fixed $\omega \neq 1$, $|y_\omega| \leq 2/|1 - \omega^2|$ on the whole half-line. The resonant response is unbounded, for example at $t = \pi/2 + 2\pi n$. Consequently, for each fixed nonresonant ω , the difference $y_\omega - y_1$ is unbounded on that half-line. Uniform convergence there is impossible, although the fixed-time limit exists. A finite-window plot cannot replace this distinction.

