

Question 6

Let $F > 0$ and $\omega \geq 0$. Consider $y'' + 2y' + 5y = F \cos(\omega t)$ on $\mathbb{R}$. For $\omega > 0$, seek a particular response $y_p = A \cos \omega t + B \sin \omega t$ and define its amplitude $R = \sqrt{A^2 + B^2}$. At $\omega = 0$, define R as the absolute value of the constant particular response.

Tasks

1. Derive A, B for $\omega > 0$ and find the constant response at zero frequency.
2. Find the frequency maximizing R on $\omega \geq 0$ and the maximum amplitude.
3. At $\omega = \sqrt{5}$, describe the phase of the response relative to the forcing and compare its amplitude with the maximum.
4. For $\omega > 0$, prove that this is the unique $2\pi/\omega$ -periodic solution and that every other solution approaches it as $t \rightarrow \infty$.

Question 6 – Solution

Strategy. Solve the two coefficient equations, then minimize the amplitude denominator instead of differentiating a complicated square root.

Step 1: Match the harmonic components. The equations are

$$(5 - \omega^2)A + 2\omega B = F, \quad -2\omega A + (5 - \omega^2)B = 0.$$

Writing $\Delta = (5 - \omega^2)^2 + 4\omega^2 > 0$, we obtain

$$\boxed{A = F(5 - \omega^2)/\Delta, \quad B = 2F\omega/\Delta.}$$

At $\omega = 0$, use the constant particular solution $F/5$.

Step 2: Maximize the amplitude. For all $\omega \geq 0$, including this constant-response convention,

$$R = \frac{F}{\sqrt{\omega^4 - 6\omega^2 + 25}} = \frac{F}{\sqrt{(\omega^2 - 3)^2 + 16}}.$$

The denominator is minimized exactly at $\omega = \sqrt{3}$, giving $\boxed{R_{\max} = F/4}$.

Step 3: Locate the quarter-cycle phase lag. At $\omega = \sqrt{5}$, $A = 0$ and $B = F/(2\sqrt{5})$, so $y_p = [F/(2\sqrt{5})] \sin(\sqrt{5}t)$. This is a cosine delayed by phase $\pi/2$. Its amplitude $F/(2\sqrt{5})$ is smaller than $F/4$; the quarter-cycle phase frequency is not the amplitude-maximizing frequency.

Step 4: Prove the steady response is unique. Every difference from y_p is $e^{-t}(C \cos 2t + D \sin 2t)$ and tends to zero forward. If a second solution had the same period $T = 2\pi/\omega$, this difference would be periodic too. For any fixed t , its value equals its value at $t + nT$, whose limit is zero. Hence the difference is identically zero, proving uniqueness of the periodic response.

