

Question 5

For $t \in \mathbb{R}$, consider

$$y'' + 2y' + 5y = e^{-t} \cos 2t, \quad y(0) = y'(0) = 0.$$

Tasks

1. Find the homogeneous roots and choose a correct resonant trial. Derive a useful operator identity with $y = e^{-t}F$.
2. Determine the particular solution and the complete zero-data response.
3. Despite the resonance, prove that every solution tends to zero as $t \rightarrow \infty$.
4. Find a simple envelope for the zero-data response and prove the bound $|y(t)| \leq 1/(4e)$ for $t \geq 0$. Is equality actually attained by the response?

Question 5 – Solution

Strategy. Resonance adds a polynomial factor, but the sign of the exponential rate still controls long-time decay.

Step 1: Identify the resonance. The roots are $-1 \pm 2i$. Thus the exponential-trigonometric forcing matches a simple complex root pair, and a suitable trial is $te^{-t}(A \cos 2t + B \sin 2t)$. Differentiation gives

$$(e^{-t}F)'' + 2(e^{-t}F)' + 5e^{-t}F = e^{-t}(F'' + 4F).$$

Step 2: Match and fit. For $F = t(A \cos 2t + B \sin 2t)$, the transformed residual is $-4A \sin 2t + 4B \cos 2t$. Hence $A = 0$, $B = 1/4$. The particular solution and its first derivative vanish at zero, so

$$y_0 = \frac{1}{4}te^{-t} \sin 2t.$$

The full family is $e^{-t}[C \cos 2t + D \sin 2t + (t/4) \sin 2t]$. Zero initial data force $C = D = 0$.

Step 3: Prove decay for every solution. For fixed C, D and $t \geq 0$, its absolute value is at most $e^{-t}(|C| + |D| + t/4)$, which tends to zero. The resonant factor t does not overcome the negative exponential rate.

Step 4: Distinguish an envelope bound from an attained maximum. We have $|y_0(t)| \leq te^{-t}/4$. The envelope attains its unique maximum at $t = 1$, since its derivative is $e^{-t}(1 - t)/4$. Thus

$$|y_0(t)| \leq 1/(4e).$$

Equality would require both $t = 1$ and $|\sin 2t| = 1$. But $|\sin 2| < 1$, so the response never attains this bound. It is a valid envelope estimate, not the exact largest displacement.

