

Question 3

Consider $y'' + y = t \cos 2t$ on $\mathbb{R}$, with $y(0) = y'(0) = 0$. A proposed trial includes only the cosine term $(at + b) \cos 2t$.

Tasks

1. Explain why a polynomial sine component must also be included, and write the smallest complete trigonometric-polynomial trial.
2. Determine its coefficients by matching both sine and cosine terms.
3. Solve the zero-data initial-value problem and verify the data.
4. Explain why nonresonance with frequency 2 does not guarantee a bounded response to this forcing. Prove that the selected response is unbounded on $[0, \infty)$.

Question 3 – Solution

Strategy. Polynomial differentiation couples the sine and cosine components even when only one appears in the forcing.

Step 1: Complete the trial space. Use

$$y_p = (at + b) \cos 2t + (ct + d) \sin 2t.$$

Frequency 2 differs from the homogeneous frequency 1, so no extra power of t is needed for resonance. Nevertheless, differentiating the proposed cosine-only term twice produces $-4a \sin 2t$, which cannot be ignored.

Step 2: Match all terms. Applying $D^2 + 1$ to the complete trial gives

$$(-3at - 3b + 4c) \cos 2t + (-3ct - 3d - 4a) \sin 2t.$$

Thus $a = -1/3$, $c = 0$, $b = 0$, $d = 4/9$, and

$$y_p = -\frac{1}{3}t \cos 2t + \frac{4}{9} \sin 2t.$$

Step 3: Fit the initial state. The particular data are $y_p(0) = 0$ and $y_p'(0) = -1/3 + 8/9 = 5/9$. The homogeneous correction is therefore $-(5/9) \sin t$, giving

$$y = -\frac{1}{3}t \cos 2t + \frac{4}{9} \sin 2t - \frac{5}{9} \sin t.$$

Its value and slope at zero vanish, and the correction leaves the verified forcing unchanged.

Step 4: Separate resonance from forcing growth. The forcing itself has a factor t and is unbounded. At $t_n = 2\pi n$, the solution is $y(t_n) = -t_n/3 \rightarrow -\infty$. At $s_n = \pi/2 + 2\pi n$, it is $y(s_n) = s_n/3 - 5/9 \rightarrow \infty$. Thus it is unbounded in both directions despite the absence of frequency resonance. Nonresonance alone is not a boundedness guarantee when the forcing amplitude grows.

