

Question 2

On $\mathbb{R}$, solve

$$y'' - 4y' + 4y = e^{2t}(6t + 2), \quad y(0) = 1, \quad y'(0) = 0.$$

Three candidate trial forms are $e^{2t}(At + B)$, $te^{2t}(At + B)$ and $t^2e^{2t}(At + B)$.

Tasks

1. Find the homogeneous roots and explain the resonance multiplicity. Derive the identity obtained by writing $y = e^{2t}F(t)$.
2. Test the first two trial forms and explain exactly why neither can produce the forcing.
3. Use the third trial to find a particular solution and then solve the initial-value problem.
4. Explain why the trial omits the constant and linear powers of F , and why their omission loses no solution freedom. Verify the final equation and data.

Question 2 – Solution

Strategy. Shift the exponential factor out of the operator; the repeated characteristic root reveals the necessary power of t .

Step 1: Identify the repeated root. The characteristic polynomial is $(r - 2)^2$. Product differentiation gives

$$(e^{2t}F)'' - 4(e^{2t}F)' + 4e^{2t}F = e^{2t}F''.$$

Thus the transformed equation is $F'' = 6t + 2$.

Step 2: Reject incomplete shifts. The first trial has $F = At + B$, so $F'' = 0$. The second has $F = At^2 + Bt$, so $F'' = 2A$, which cannot supply the linear term $6t$. Multiplying the usual degree-one trial by only one power of t is still insufficient at a double root.

Step 3: Use the correct trial. With $F = At^3 + Bt^2$, coefficient matching gives $6At + 2B = 6t + 2$, hence $A = B = 1$. The complete solution is

$$y = e^{2t}(C + Dt + t^2 + t^3).$$

The initial value sets $C = 1$, and the slope at zero is $2C + D$, so $D = -2$. Therefore

$$y = e^{2t}(1 - 2t + t^2 + t^3).$$

Step 4: Account for every constant. Constant and linear terms in F have zero second derivative and correspond exactly to e^{2t} and te^{2t} , the homogeneous modes. Leaving them out of the particular trial avoids redundant coefficients; the constants C, D restore them in the general solution. For the final polynomial, $F'' = 2 + 6t$, so the equation holds. Also $F(0) = 1$ and $2F(0) + F'(0) = 2 - 2 = 0$, verifying both data.