

Question 10

On $0 \leq t \leq 1$, design a forcing g for $y'' + 2y' + y = g(t)$ so that the trajectory has

$$y(0) = 0, \quad y'(0) = 0, \quad y(1) = 1, \quad y'(1) = 0.$$

Restrict the desired trajectory to polynomials of degree at most three.

Tasks

1. Find the unique polynomial meeting the four endpoint requirements.
2. Derive the forcing g that makes this polynomial an exact solution and verify all four requirements.
3. Prove that this forcing changes sign exactly once on $[0, 1]$, locating the zero between $3/4$ and $4/5$. Can a nonnegative forcing realize this same cubic trajectory?
4. Keep the designed forcing but perturb the initial data to $y(0) = \varepsilon$, $y'(0) = 0$. Find the exact position and slope errors at $t = 1$, and determine whether the target state is still reached when $\varepsilon \neq 0$.

Question 10 – Solution

Strategy. Prescribe a trajectory and compute its forcing directly; then use a homogeneous difference equation to analyze initial-data error.

Step 1: Fit the cubic trajectory. Write $y = c_0 + c_1t + c_2t^2 + c_3t^3$. The initial conditions give $c_0 = c_1 = 0$. The final conditions give $c_2 + c_3 = 1$ and $2c_2 + 3c_3 = 0$, so

$$y_*(t) = 3t^2 - 2t^3.$$

The coefficient equations have a unique solution, including among all polynomials of lower degree.

Step 2: Compute the forcing. The derivatives are $y'_* = 6t - 6t^2$ and $y''_* = 6 - 12t$. Thus

$$g = y''_* + 2y'_* + y_* = 6 - 9t^2 - 2t^3.$$

The derivative formulas give zero initial value and slope, and final value $3 - 2 = 1$ with slope $6 - 6 = 0$. The computed forcing verifies the equation identically.

Step 3: Locate the sign change. We have $g(0) = 6$, $g(1) = -5$, and $g' = -18t - 6t^2 < 0$ for $t > 0$. Therefore there is exactly one zero. More precisely,

$$g(3/4) = 3/32 > 0, \quad g(4/5) = -98/125 < 0,$$

so the zero lies strictly between $3/4$ and $4/5$. This same cubic fixes g pointwise, so a nonnegative forcing cannot realize it on the whole interval. This conclusion concerns the specified cubic, not every possible trajectory.

Step 4: Propagate the initial error. The difference $h = y - y_*$ satisfies $h'' + 2h' + h = 0$, with $h(0) = \varepsilon$, $h'(0) = 0$. Hence $h = \varepsilon(1 + t)e^{-t}$ and $h' = -\varepsilon te^{-t}$. At the endpoint,

$$y(1) - 1 = 2\varepsilon/e, \quad y'(1) = -\varepsilon/e.$$

For nonzero ε the target state is missed. The forcing is unchanged; the error evolves entirely through the homogeneous correction.