

Question 9

Let y solve $y'' + y = 1$ with $y(0) = y'(0) = 0$. On $0 \leq t \leq 1$, consider the approximation $Y(t) = t^2/2$ and the error $E = Y - y$. You may use $0 \leq \sin r \leq r$ for $0 \leq r \leq 1$.

Tasks

1. Compute the residual $R = Y'' + Y - 1$ and derive the differential equation and initial data satisfied by E .
2. Verify by differentiating twice that $\int_0^t \sin(t-s)s^2/2 \, ds$ solves this error initial-value problem. Justify that it equals E .
3. Prove the rigorous error bound $0 \leq E(t) \leq t^4/24$ on $[0, 1]$, including the direction of the approximation error.
4. Find the exact solution y by verification, compute the actual error at 1 to six decimal places, and compare it with the bound. Why does a residual vanishing only at zero not prove exactness?

Question 9 – Solution

Strategy. Treat the approximation residual as a forcing for the error, then bound the resulting integral with a positive kernel.

Step 1: Form the error equation. Since $Y'' = 1$, the residual is $R = t^2/2$. Subtraction gives

$$E'' + E = t^2/2, \quad E(0) = E'(0) = 0.$$

Step 2: Verify the integral. Set $F(t) = \int_0^t \sin(t-s)s^2/2 \, ds$. Leibniz differentiation gives

$$F' = \int_0^t \cos(t-s)s^2/2 \, ds, \quad F'' = t^2/2 - \int_0^t \sin(t-s)s^2/2 \, ds.$$

Thus $F'' + F = t^2/2$ and $F(0) = F'(0) = 0$. The continuous coefficients and forcing give uniqueness, so $F = E$.

Step 3: Bound the error. For $0 \leq s \leq t \leq 1$, the supplied sine inequality gives

$$0 \leq E(t) \leq \frac{1}{2} \int_0^t (t-s)s^2 \, ds = \frac{1}{2}(t^4/3 - t^4/4) = \boxed{t^4/24}.$$

Hence Y overestimates the solution. The bound vanishes at zero and remains valid across the entire stated interval.

Step 4: Check against the exact response. The function $y = 1 - \cos t$ has the required equation and zero data. Therefore

$$E(1) = \cos 1 - \frac{1}{2} \approx 0.040302 < 1/24 \approx 0.041667.$$

The residual is zero at the initial point but is positive elsewhere on $(0, 1]$. A solution must satisfy its equation at every point, so agreement at one point and in the initial data does not make Y exact.

