

Question 8

Let u, v satisfy the same forced equation $y'' + p(t)y' + q(t)y = g(t)$ on a regular interval, and write $W = uv' - u'v$. For a concrete example on $\mathbb{R}$, take

$$u = 1 - \cos t, \quad v = 1 - \cos t + \sin t.$$

Tasks

1. Derive the differential identity for W when the common forcing is nonzero.
2. Verify that the example functions solve $y'' + y = 1$. Compute their Wronskian and explain why its zero at zero does not force it to vanish everywhere.
3. Prove that the two example functions are linearly independent as functions, and identify the homogeneous solution obtained from their difference.
4. Classify the constant affine combinations of u, v that solve the same forced equation. Which initial values and slopes at zero can they attain, and do they give the entire forced solution set?

Question 8 – Solution

Strategy. Recompute the Wronskian identity with the forcing included, then distinguish linear independence from a homogeneous fundamental set.

Step 1: Keep the forcing terms. Differentiating and substituting gives

$$W' = uv'' - u''v = u(g - pv' - qv) - v(g - pu' - qu),$$

so

$$W' = -pW + g(u - v).$$

The extra term is absent from Abel's homogeneous identity and generally does not vanish.

Step 2: Verify the example. We have $u'' + u = \cos t + 1 - \cos t = 1$, and adding $\sin t$ preserves this forcing. Direct computation gives

$$W = (1 - \cos t) \cos t - \sin^2 t = \cos t - 1.$$

It is zero at 0 but equals -2 at π . Its derivative is $-\sin t$, matching $g(u - v) = -\sin t$. The homogeneous zero-or-never-zero conclusion is inapplicable to this forced pair.

Step 3: Test independence. If $Au + Bv = 0$, evaluation at π gives $2A + 2B = 0$ and at $\pi/2$ gives $A + 2B = 0$. Hence $A = B = 0$. The difference $v - u = \sin t$ solves the associated homogeneous equation. Independence of the forced functions does not make them a fundamental set of homogeneous solutions.

Step 4: Describe their affine family. The forcing of $Au + Bv$ is $A + B$, so retaining forcing 1 requires $A + B = 1$. Such combinations are

$$u + B \sin t = 1 - \cos t + B \sin t.$$

They have initial value zero and arbitrary slope B . They give exactly the forced solutions with initial value zero, but miss every solution with a different initial value. The full family is $1 + C \cos t + D \sin t$, with two independent initial-data freedoms.