

Question 7

On $t > 0$, consider

$$y'' = 1/t, \quad y(1) = 0, \quad y'(1) = 0,$$

with supplied particular solution $p(t) = t \ln t - t$. The associated homogeneous equation has the globally smooth solutions $1, t$.

Tasks

1. Verify the particular solution and find every solution on $(0, \infty)$.
2. Solve the initial-value problem and determine its monotonicity, convexity and minimum.
3. Find the right-hand limit at zero and decide whether the selected solution admits continuous, C^1 , or C^2 extension through zero.
4. Can any homogeneous correction to p produce a C^1 extension through zero? Explain why smooth homogeneous solutions do not guarantee continuation of the forced problem.

Question 7 – Solution

Strategy. The forcing can impose endpoint behavior that no homogeneous correction can cancel.

Step 1: Verify and complete. Since $p' = \ln t$ and $p'' = 1/t$, it is a particular solution. Adding all affine homogeneous functions gives

$$y = t \ln t - t + At + B \quad (t > 0).$$

Step 2: Fit and analyze. The slope at 1 gives $A = 0$, and the value then gives $B = 1$. Thus

$$y = t \ln t - t + 1, \quad y' = \ln t, \quad y'' = 1/t > 0.$$

The function decreases on $(0, 1)$, increases on $(1, \infty)$, and is strictly convex. Its unique global minimum is $y(1) = 0$.

Step 3: Test extension of the selected function. Since $t \ln t \rightarrow 0$ as $t \downarrow 0$, we have $y(t) \rightarrow 1$. Setting $y(0) = 1$ and, for example, $y(t) = 1$ for $t < 0$ gives a continuous extension. A finite derivative at zero is impossible because

$$\frac{y(t) - 1}{t} = \ln t - 1 \rightarrow -\infty.$$

Hence there is no C^1 extension and therefore no C^2 extension. The limiting point in the graph is excluded from the original domain.

Step 4: Test every correction. For general A, B , continuity would require $y(0) = B$, but the difference quotient is $\ln t - 1 + A$, again tending to $-\infty$. No finite homogeneous constants cure this. Although $1, t$ extend smoothly, the forcing $1/t$ is undefined at zero. The maximal regular interval containing the initial point is $(0, \infty)$, not all of $\mathbb{R}$.

