

Question 6

For $y'' + y = \cos t$ on $\mathbb{R}$, a proposed particular solution is $p(t) = \frac{1}{2}t \sin t$. The forcing is bounded in absolute value by one.

Tasks

1. Verify the particular solution and find the full solution family.
2. Solve the initial-value problem with arbitrary data $y(0) = a$, $y'(0) = b$. Identify the zero-data response.
3. Prove that every solution is unbounded above and below on $[0, \infty)$, using two explicit sequences of times. Can a homogeneous correction remove this behavior?
4. For the zero-data response, give exact upper and lower envelope bounds on $t \geq 0$ and the times when they are attained. Explain why envelope contacts need not be stationary points.

Question 6 – Solution

Strategy. Verify the supplied growing response and use exact phase samples to rule out cancellation by bounded homogeneous terms.

Step 1: Verify and complete. Differentiation gives

$$p' = \frac{1}{2} \sin t + \frac{1}{2} t \cos t, \quad p'' = \cos t - \frac{1}{2} t \sin t.$$

Thus $p'' + p = \cos t$ and the general solution is

$$y = A \cos t + B \sin t + \frac{1}{2} t \sin t.$$

Step 2: Fit the data. Since $p(0) = p'(0) = 0$, the data give $A = a$, $B = b$. In particular the zero-data response is exactly $y_0 = \frac{1}{2} t \sin t$.

Step 3: Prove both unbounded excursions. At $t_n = \pi/2 + 2\pi n$, $n = 0, 1, \dots$, we have $y(t_n) = b + t_n/2 \rightarrow \infty$. At $s_n = 3\pi/2 + 2\pi n$, we have $y(s_n) = -b - s_n/2 \rightarrow -\infty$. These values do not depend on a . Thus no constant homogeneous correction removes the unbounded behavior, even though the forcing is bounded.

Step 4: Distinguish bounds from extrema. On $t \geq 0$,

$$-t/2 \leq y_0(t) \leq t/2.$$

For $t > 0$, the upper envelope is attained exactly at t_n and the lower at s_n ; both bounds also coincide with y_0 at zero. At a positive upper contact $y'_0 = 1/2$, and at a lower contact $y'_0 = -1/2$. These slopes match those of the respective envelopes and are not zero, so the contacts are not stationary points of y_0 .

