

Question 5

For $y'' - y = 2 \cos t$ on $\mathbb{R}$, a proposed particular solution is $p = -\cos t$. Prescribe real initial data $y(0) = a$, $y'(0) = b$.

Tasks

1. Verify the particular solution and find the full initial-value solution in terms of a, b .
2. Find a necessary and sufficient relation between a, b for boundedness on $[0, \infty)$. For those data, determine the limiting difference $y(t) - p(t)$.
3. Find the unique solution bounded on all of $\mathbb{R}$. Does the selected bounded solution necessarily have a limit as $t \rightarrow \infty$?
4. Compare the trajectories from $(a, b) = (0, -1)$ and $(0, 0)$. Explain why bounded forcing alone does not make every solution bounded.

Question 5 – Solution

Strategy. The difference from a particular solution follows the homogeneous equation, whose growing mode determines boundedness.

Step 1: Verify and fit. Since $p'' = \cos t$, $p'' - p = 2 \cos t$. The general solution is $-\cos t + Ae^t + Be^{-t}$. The data give $A + B = a + 1$, $A - B = b$, hence

$$y = -\cos t + \frac{a+1+b}{2}e^t + \frac{a+1-b}{2}e^{-t}.$$

Step 2: Eliminate the growing mode. The bounded oscillatory and decaying terms cannot cancel a nonzero Ae^t as $t \rightarrow \infty$. Thus forward boundedness holds exactly when

$$b = -a - 1.$$

In that case $y = -\cos t + (a+1)e^{-t}$, so $y - p \rightarrow 0$. This convergence is to a time-varying particular solution, not necessarily to a constant.

Step 3: Require both time directions. Boundedness as $t \rightarrow -\infty$ also forces $B = 0$. Thus the unique globally bounded solution is $y = -\cos t$, with data $(-1, 0)$. It has no limit at positive infinity: its values at $2\pi n$ and $(2n+1)\pi$ are -1 and 1 . The forward-bounded family also has these two limiting subsequences because its decaying correction tends to zero.

Step 4: Compare the selected data. Data $(0, -1)$ give $y_s = -\cos t + e^{-t}$, which is bounded forward. Data $(0, 0)$ give $y_u = -\cos t + \cosh t$, which is unbounded forward. Both have the same bounded forcing; the second retains a growing homogeneous component. The finite plot illustrates the distinction; the mode argument proves the global claim.

