

Question 4

For the operator $L[y] = y'' + y$, consider the supplied responses

$$u(t) = \frac{\cos t - \cos 2t}{3}, \quad v(t) = 2(t - \sin t)$$

on $\mathbb{R}$.

Tasks

1. Verify that $L[u] = \cos 2t$, $L[v] = 2t$, and that both responses have zero value and slope at zero.
2. Construct the solution of $y'' + y = \cos 2t + 2t$ with $y(0) = 1$, $y'(0) = -2$. Verify the data.
3. State and prove how both forcing and initial data combine when a constant linear combination of two responses is formed.
4. Let $r = u + \cos t$ and $s = v + \cos t$. Each now has data $(1, 0)$. Correct the naive sum $r + s$ so that it has the combined forcing but only the single prescribed data pair $(1, 0)$.

Question 4 – Solution

Strategy. Superposition combines the initial data as well as the forcing; use a homogeneous correction to achieve the desired data.

Step 1: Verify the supplied responses. We have $u'' = (-\cos t + 4 \cos 2t)/3$, so $u'' + u = \cos 2t$. Also $v' = 2(1 - \cos t)$ and $v'' = 2 \sin t$, so $v'' + v = 2t$. At zero both values and both slopes vanish.

Step 2: Add the forcing and fit the state. The sum $u + v$ handles the combined forcing with zero initial data. Adding the homogeneous solution $\cos t - 2 \sin t$ gives

$$y = u + v + \cos t - 2 \sin t = \frac{4}{3} \cos t - \frac{1}{3} \cos 2t + 2t - 4 \sin t.$$

At zero the value is $4/3 - 1/3 = 1$ and the slope is $2 - 4 = -2$. Linearity verifies the combined forcing, and regular uniqueness proves this is the initial-value solution.

Step 3: Track the full linear map. If $L[y_j] = g_j$ and the initial data are (a_j, b_j) , then $Ay_1 + By_2$ has forcing $Ag_1 + Bg_2$ and data

$$(Aa_1 + Ba_2, Ab_1 + Bb_2).$$

This follows directly from linearity of L , evaluation and differentiation. Ignoring the data part can produce the right equation with the wrong initial state.

Step 4: Remove duplicated initial data. The naive sum $r + s$ has data $(2, 0)$. Subtract one copy of the homogeneous solution $\cos t$:

$$r + s - \cos t = u + v + \cos t.$$

The forcing is unchanged, while the data become $(1, 0)$. This correction differs from Step 2 because the required initial slope here is zero, not -2 .