

Question 1

Let $L[y] = y'' + p(t)y' + q(t)y$ on a connected open interval I , where p, q, g are continuous. Suppose y_p is one solution of $L[y] = g$, and u, v form a fundamental set for $L[y] = 0$.

Tasks

1. Prove that every solution of $L[y] = g$ has the form $y_p + Au + Bv$, with unique constants A, B . Prove both directions.
2. If initial data $y(t_0) = a, y'(t_0) = b$ are prescribed, state the initial data that the homogeneous correction must satisfy and explain uniqueness.
3. For $y'' - 3y' + 2y = 2t^2 - 6t + 2$ on $\mathbb{R}$, verify the supplied particular solution $y_p = t^2$ and find the general solution.
4. Solve the example with $y(0) = 1, y'(0) = 0$, verifying the equation and initial data.

Question 1 – Solution

Strategy. Subtract one known forced solution so that the remaining unknown satisfies a homogeneous equation.

Step 1: Prove the representation. If $L[y] = L[y_p] = g$, linearity gives $L[y - y_p] = 0$. Completeness of u, v then gives $y - y_p = Au + Bv$. Conversely, $L[y_p + Au + Bv] = g + A(0) + B(0) = g$. Independence makes the two constants unique for each solution. Thus

$$y = y_p + Au + Bv.$$

Step 2: Transfer the initial data. The correction $h = y - y_p$ must satisfy

$$h(t_0) = a - y_p(t_0), \quad h'(t_0) = b - y_p'(t_0).$$

Regular homogeneous existence and uniqueness determine h on I . Therefore the forced initial-value problem is unique too. The coefficients in the correction must fit these adjusted data, not automatically the original data.

Step 3: Verify and complete the example. For $y_p = t^2$, the left-hand side is $2 - 6t + 2t^2$, as required. The characteristic polynomial of the associated homogeneous equation is $(r - 1)(r - 2)$, so

$$y = t^2 + Ae^t + Be^{2t}.$$

The exponentials are independent homogeneous solutions, and the representation theorem proves that no further solution freedom is missing.

Step 4: Fit and check. At zero, the equations are $A + B = 1$ and $A + 2B = 0$. They give $A = 2$, $B = -1$, hence

$$y = t^2 + 2e^t - e^{2t}.$$

Its initial value is $2 - 1 = 1$ and its slope is $0 + 2 - 2 = 0$. The polynomial contributes the entire forcing, while each exponential has zero residual, verifying the differential equation everywhere.