

Question 10

Let u, v be C^2 real functions on an interval and let θ be a real C^2 function. Define time-dependent combinations

$$f = u \cos \theta + v \sin \theta, \quad g = -u \sin \theta + v \cos \theta.$$

At every time the two-by-two coefficient matrix has determinant 1.

Tasks

1. Derive the actual expression for $W[f, g]$ in terms of u, v, θ and their derivatives. Explain why the constant change-of-basis formula cannot simply be used.
2. Take $u = \cos t$, $v = \sin t$, $\theta = t$ on $\mathbb{R}$. Compute f, g and their Wronskian. Does the pointwise invertibility preserve a fundamental set for $y'' + y = 0$?
3. Keep the same u, v but take $\theta = t/2$. Show that a nonzero resulting Wronskian still does not make the new pair solutions of the original equation.
4. For these trigonometric u, v , classify every real C^2 function θ on $\mathbb{R}$ for which both f, g solve $y'' + y = 0$.

Question 10 – Solution

Strategy. Differentiate the moving coefficients too; pointwise invertibility is not a constant linear transformation of solution functions.

Step 1: Include the extra derivative terms. In the derivatives of f, g , the terms involving u', v' contribute $W[u, v]$, as in a constant rotation. The remaining terms are $\theta'g$ in f' and $-\theta'f$ in g' . Hence

$$W[f, g] = W[u, v] - \theta'(f^2 + g^2) = W[u, v] - \theta'(u^2 + v^2).$$

The additional term is exactly what the constant-coefficient formula omits.

Step 2: Observe a complete collapse. For $u = \cos t, v = \sin t$, the addition formulas give $f = \cos(t - \theta)$, $g = \sin(t - \theta)$. At $\theta = t$, $f = 1$ and $g = 0$, so $W = 0$. They are dependent, and $f'' + f = 1 \neq 0$. Pointwise invertibility of the moving coefficient matrix does not preserve the solution space.

Step 3: Test a nonzero determinant. At $\theta = t/2$, the functions are $\cos(t/2), \sin(t/2)$ and $W = 1/2$. Each has $y'' + y = (3/4)y$, so neither solves the original equation identically. They instead form a fundamental set for $y'' + y/4 = 0$.

Step 4: Classify all valid rotations. Put $\varphi = t - \theta$. The residuals for $f = \cos \varphi, g = \sin \varphi$ are

$$(1 - \varphi'^2) \cos \varphi - \varphi'' \sin \varphi, \quad (1 - \varphi'^2) \sin \varphi + \varphi'' \cos \varphi.$$

Both vanish exactly when $\varphi'^2 = 1$ and $\varphi'' = 0$, since the displayed rotation is invertible. On the connected real line this gives $\varphi = \pm t + C$, hence

$$\theta(t) = C \quad \text{or} \quad \theta(t) = 2t + C, \quad C \in \mathbb{R}.$$

In the second case the final functions are still constant combinations of $\cos t, \sin t$, despite their time-dependent defining coefficients.