

Question 8

Suppose p, q are real continuous functions on $\mathbb{R}$, each periodic with period $T > 0$. Consider $y'' + p(t)y' + q(t)y = 0$. A T -periodic solution satisfies $y(t + T) = y(t)$ for every real t .

Tasks

1. Prove that the existence of two independent T -periodic solutions requires $\int_0^T p(t) dt = 0$.
2. If $p \geq 0$ and p is not identically zero, decide whether two such periodic solutions can exist.
3. Is the zero-integral condition sufficient? Analyze $y'' - y = 0$ as a counterexample and justify the absence of nonzero periodic solutions.
4. Does the existence of just one nonzero T -periodic solution force the same integral condition? Analyze $y'' + y' = 0$ and identify a nonperiodic companion.

Question 8 – Solution

Strategy. Periodicity forces the Wronskian to return to its starting value, while Abel's identity specifies its multiplicative change.

Step 1: Compare one full period. Differentiating $y(t + T) = y(t)$ shows that the slope is periodic too. Thus two periodic solutions have $W(T) = W(0)$. Independence gives $W(0) \neq 0$, so Abel's identity implies

$$\exp\left[-\int_0^T p(t) dt\right] = 1.$$

The integral is real; consequently

$$\int_0^T p(t) dt = 0.$$

Step 2: Exclude nontrivial nonnegative damping. By continuity and periodicity, if $p \geq 0$ and is not identically zero, it is positive on some interval within a period. Hence its integral over a period is strictly positive. The necessary condition fails, so two independent T -periodic solutions cannot exist.

Step 3: Disprove sufficiency. For $y'' - y = 0$, $p = 0$ satisfies the integral condition for every T . Every solution is $Ae^t + Be^{-t}$. A continuous periodic function is bounded on $\mathbb{R}$; boundedness as $t \rightarrow \infty$ forces $A = 0$, and boundedness as $t \rightarrow -\infty$ forces $B = 0$. Thus there are no nonzero periodic solutions, much less two independent ones.

Step 4: Test a single periodic solution. In $y'' + y' = 0$, the constant solution $u = 1$ is nonzero and periodic for every $T > 0$, while $p = 1$ has integral $T \neq 0$. A companion is $v = e^{-t}$, with $W[u, v] = -e^{-t} \neq 0$; it is not periodic. One periodic solution does not make a nonzero pair Wronskian periodic and imposes no such zero-integral requirement.