

Question 7

Let p be C^1 and q continuous on an open interval I containing zero, and set $P(t) = \int_0^t p(s) ds$. Consider the substitution $y = e^{-P/2}z$ in $y'' + py' + qy = 0$.

Tasks

1. Derive the differential equation satisfied by z and show that its first-derivative term vanishes.
2. For two transformed solutions, derive the relation between their Wronskian and the original pair's Wronskian. Explain why the transformed Wronskian is constant.
3. Apply the method to $y'' + 2ty' + (t^2 + 3)y = 0$ on $\mathbb{R}$. Construct a fundamental pair normalized to data $(1, 0)$ and $(0, 1)$ at zero and compute its Wronskian.
4. Solve $y(0) = 1$, $y'(0) = 2$ and explain why both members of your fundamental pair tending to zero as $t \rightarrow \infty$ does not contradict independence.

Question 7 – Solution

Strategy. A common nonzero multiplier changes the Wronskian by its square and can remove the first-derivative coefficient.

Step 1: Differentiate the substitution. Put $h = e^{-P/2}$, so $h'/h = -p/2$ and $h''/h = -p'/2 + p^2/4$. Substitution of $y = hz$ gives

$$z'' + (2h'/h + p)z' + (h''/h + ph'/h + q)z = 0,$$

and therefore

$$z'' + [q - p'/2 - p^2/4]z = 0.$$

Step 2: Transform the determinant. Expanding derivatives cancels the terms containing hh' :

$$W[hz_1, hz_2] = h^2 W[z_1, z_2] = e^{-P} W[z_1, z_2].$$

The transformed equation has no first-derivative term, so its Wronskian has derivative zero. The nonzero multiplier preserves independence.

Step 3: Solve the transformed example. Here $P = t^2$ and $q - p'/2 - p^2/4 = t^2 + 3 - 1 - t^2 = 2$. Thus $z'' + 2z = 0$, and the normalized pair is

$$\phi = e^{-t^2/2} \cos(\sqrt{2}t), \quad \psi = e^{-t^2/2} \frac{\sin(\sqrt{2}t)}{\sqrt{2}}.$$

At zero, $h = 1$ and $h' = 0$, so the data remain $(1, 0)$ and $(0, 1)$. The transformed Wronskian is 1, giving $W[\phi, \psi] = e^{-t^2} > 0$ everywhere.

Step 4: Match and interpret. The normalized data give

$$y = \phi + 2\psi = e^{-t^2/2} [\cos(\sqrt{2}t) + \sqrt{2} \sin(\sqrt{2}t)].$$

Its value and slope at zero are 1, 2. Both basis functions decay to zero because their trigonometric factors are bounded and the Gaussian factor decays. Dependence would require an exact constant relation on the interval; the strictly positive finite-time Wronskian rules that out.