

Question 6

Let u, v be independent real solutions of $y'' + p(t)y' + q(t)y = 0$ on an open interval I , with continuous coefficients. Suppose $a < b$ lie in I and are consecutive zeros of u : $u(a) = u(b) = 0$ and $u(t) \neq 0$ for $a < t < b$.

Tasks

1. Prove that the zeros at a, b are simple and that $v(a), v(b)$ are nonzero.
2. Use the sign of the Wronskian at the endpoints to prove that $v(a)$ and $v(b)$ have opposite signs.
3. Differentiate v/u on (a, b) and use it to prove that v has exactly one zero there.
4. Apply the result to $u = \sin t$, $v = \cos t + 2 \sin t$ between 0 and π , verifying independence and finding the unique zero explicitly.

Question 6 – Solution

Strategy. Use the constant sign of the Wronskian first at the endpoints and then in the derivative of a ratio.

Step 1: Exclude repeated and shared zeros. If u and u' both vanished at an endpoint, regular uniqueness would make u identically zero, contrary to independence. Thus both zeros are simple. Since W never vanishes, $W(a) = -u'(a)v(a)$ and $W(b) = -u'(b)v(b)$ force $v(a), v(b) \neq 0$.

Step 2: Compare endpoint signs. Multiply u by -1 if necessary so that $u > 0$ on (a, b) . Simplicity then implies $u'(a) > 0$ and $u'(b) < 0$. Abel's identity says $W(a), W(b)$ have the same sign. The two endpoint formulas therefore force $v(a), v(b)$ to have opposite signs. Continuity gives at least one zero of v inside.

Step 3: Prove uniqueness of that zero. On the zero-free interval for u ,

$$(v/u)' = \frac{uv' - u'v}{u^2} = \frac{W}{u^2}.$$

The derivative has a fixed nonzero sign, so the ratio is strictly monotone. Its zeros are precisely the zeros of v there. It can vanish at most once; combined with the preceding existence argument, this proves exactly one zero.

Step 4: Compute the example. Both candidates solve $y'' + y = 0$, and

$$W[\sin t, \cos t + 2 \sin t] = -1.$$

The endpoint values of v are $1, -1$. In $(0, \pi)$, the equation $\cos t + 2 \sin t = 0$ has its root in the second quadrant, with

$$t = \pi - \arctan(1/2).$$

The separation argument proves there are no other roots between these consecutive zeros of $\sin t$.

