

Question 5

Two independent solutions of a regular equation $y'' + p(t)y' + q(t)y = 0$ on $\mathbb{R}$ have measured ordered Wronskians

$$W(0) = 6, \quad W(2) = 3, \quad W(5) = 12.$$

Assume p, q are continuous and the measurements are exact.

Tasks

1. Find $\int_0^2 p(t) dt$ and $\int_2^5 p(t) dt$, and the average value of p on each interval.
2. Prove that p is positive somewhere in $(0, 2)$, negative somewhere in $(2, 5)$, and zero somewhere in $(0, 5)$.
3. Under the additional assumption that $p(t) = a + bt$, determine a, b and the full Wronskian function. Check all three measurements.
4. For this affine model, find the unique minimum of W on $[0, 5]$. Explain why three Wronskian samples alone do not prove that the true coefficient is affine.

Question 5 – Solution

Strategy. Ratios of Wronskian measurements give integrals of the coefficient, rather than its pointwise values.

Step 1: Extract the averages. Abel's identity gives

$$\int_0^2 p = \ln 2, \quad \int_2^5 p = -\ln 4.$$

The average values are $\boxed{(\ln 2)/2}$ and $\boxed{-(\ln 4)/3}$ respectively.

Step 2: Force a sign change. A continuous function with positive integral must be positive at an interior point; likewise the negative integral requires an interior negative value. Choosing one point from each interval, the intermediate value theorem supplies a zero between them. This conclusion does not specify the value of p at time 2.

Step 3: Fit the affine model. The coefficient equations are $2a + 2b = \ln 2$ and $3a + (21/2)b = -2\ln 2$. Solving gives

$$p(t) = \frac{29 - 14t}{30} \ln 2, \quad \boxed{W(t) = 6 \cdot 2^{(7t^2 - 29t)/30}}.$$

At $t = 0, 2, 5$, the exponents are 0, -1 , 1, giving 6, 3, 12 as required.

Step 4: Locate and qualify the minimum. Since $W' = -pW$ and $W > 0$, the Wronskian decreases until $t = 29/14$ and increases afterward. Its unique minimum on $[0, 5]$ is

$$\boxed{W(29/14) = 6 \cdot 2^{-841/840}}.$$

Affineness is an extra modeling assumption. For example, adding any multiple of $\cos(2\pi t)$ to this p leaves both measured integrals unchanged, but generally gives a nonaffine coefficient and a different Wronskian between the measurement times.

