

Question 4

A regular equation $y'' + p(t)y' + q(t)y = 0$ on $\mathbb{R}$ has a known solution $u = e^t$. A desired companion v must satisfy

$$W[u, v](t) = 2e^{-t^2}, \quad v(0) = 0.$$

Assume p, q are continuous; an unevaluated definite integral is an acceptable exact answer.

Tasks

1. Determine the only possible coefficients p, q .
2. Construct v using the prescribed Wronskian and initial value, and find $v'(0)$.
3. Verify both the Wronskian and the differential equation without evaluating the integral.
4. If the known solution u were not supplied, would the prescribed Wronskian alone determine q ? Justify your answer using initial-data normalization for arbitrary continuous q .

Question 4 – Solution

Strategy. Abel's identity determines the first-derivative coefficient; the known nonzero solution then determines the remaining coefficient.

Step 1: Recover the equation. Since W never vanishes,

$$p = -W'/W = 2t.$$

Substituting $u = e^t$ gives $1 + p + q = 0$, hence

$$y'' + 2ty' - (1 + 2t)y = 0.$$

Step 2: Recover the companion. The identity $W = e^t(v' - v)$ gives $v' - v = 2e^{-t^2-t}$. Multiplying by e^{-t} and integrating from zero yields

$$v(t) = e^t \int_0^t 2e^{-s^2-2s} ds.$$

The integral is smooth on all of $\mathbb{R}$. It gives $v(0) = 0$ and $v'(0) = 2$.

Step 3: Verify by differentiation. Write J for the integral, so $J' = 2e^{-t^2-2t}$. Then $v' = v + 2e^{-t^2-t}$ and $e^t(v' - v) = 2e^{-t^2}$, as prescribed. If $L[y] = y'' + py' + qy$, direct expansion gives

$$uL[v] - vL[u] = W' + pW.$$

Here $L[u] = 0$ and $W' + 2tW = 0$. Since $u = e^t \neq 0$, this proves $L[v] = 0$ everywhere.

Step 4: Identify what the Wronskian omits. With $p = 2t$ and any continuous choice of q , regular existence supplies solutions with data $(1, 0)$ and $(0, 2)$ at zero. Their initial Wronskian is 2, so Abel's identity gives the same $2e^{-t^2}$ for every such q . Thus the Wronskian alone fixes p but not q ; the supplied nonzero seed was essential for the latter.