

Question 3

Consider the functions $u(t) = t^3$ and $v(t) = |t|^3$ on $\mathbb{R}$. A student claims that a Wronskian which is identically zero always proves linear dependence of two C^2 functions.

Tasks

1. Prove that both functions are C^2 on $\mathbb{R}$, explicitly checking the derivatives of v at zero.
2. Compute $W[u, v]$ everywhere and determine whether the two functions are linearly independent on $\mathbb{R}$.
3. Explain precisely why the common regular-equation theorem does not contradict your result. Could both functions solve a normalized equation with continuous coefficients on $\mathbb{R}$?
4. Verify that both solve the undivided equation $t^2 y'' - 2ty' = 0$ on $\mathbb{R}$. Compare the ratio v/u on the two sides of zero.

Question 3 – Solution

Strategy. Check the counterexample directly, then identify the hypothesis missing from the student's assertion.

Step 1: Check regularity. For $t \neq 0$, $v' = 3t|t|$ and $v'' = 6|t|$. At zero,

$$v'(0) = \lim_{h \rightarrow 0} |h|^3/h = 0, \quad v''(0) = \lim_{h \rightarrow 0} 3h|h|/h = 0.$$

These agree continuously with the formulas away from zero, so v is C^2 ; u is a polynomial and is smooth.

Step 2: Compare the Wronskian and independence. On $t > 0$, $v = u$; on $t < 0$, $v = -u$. Thus $W = 0$ on each side, and at zero both data columns are zero, so $\boxed{W \equiv 0}$. Yet $Au + Bv = 0$ on all of $\mathbb{R}$ forces $A + B = 0$ on the positive side and $A - B = 0$ on the negative side. Hence $A = B = 0$: the functions are independent on $\mathbb{R}$.

Step 3: Restore the missing hypothesis. The zero-Wronskian implication requires solutions of one common regular homogeneous linear equation on a connected interval, not merely two C^2 functions. The nonzero solution u would have zero value and slope at zero, contradicting regular uniqueness. Thus no normalized equation with continuous coefficients on $\mathbb{R}$ can have both as solutions.

Step 4: Verify a singular equation. For u , the residual is $t^2(6t) - 2t(3t^2) = 0$. On each side $v = \pm u$, so its residual also vanishes; at zero both terms vanish. The leading coefficient is zero there. The ratio v/u is 1 on $t > 0$ and -1 on $t < 0$, and is undefined at the shared zero. Local constant ratios need not join into one global constant.

