

Question 1

For two real solutions u, v of $y'' + p(t)y' + q(t)y = 0$ on a connected open interval I , define the ordered Wronskian

$$W[u, v](t) = u(t)v'(t) - u'(t)v(t).$$

Assume p, q are continuous and $t_0 \in I$.

Tasks

1. Derive a first-order equation for W and solve it in terms of $W(t_0)$. Your derivation must include the case $W(t_0) = 0$.
2. Prove that W either vanishes everywhere on I or never vanishes there. Explain the consequences for independence of the two solutions.
3. For $(1 + t^2)y'' + 2ty' + (1 + t^2)y = 0$ on $\mathbb{R}$, find $W(t)$ if $W(0) = -3$, without solving the second-order equation.
4. Find the sign and limiting values of this Wronskian. Does its approach to zero at infinity imply that the solutions become dependent at a finite time?

Question 1 – Solution

Strategy. Differentiate the determinant and substitute the common equation before integrating.

Step 1: Derive Abel's identity. Differentiation cancels the mixed terms:

$$W' = uv'' - u''v = u(-pv' - qv) - (-pu' - qu)v = -pW.$$

Multiplication by $\exp(\int_{t_0}^t p(s) ds)$ gives a function with zero derivative. Thus, without dividing by W ,

$$W(t) = W(t_0) \exp\left[-\int_{t_0}^t p(s) ds\right].$$

This includes $W \equiv 0$ when $W(t_0) = 0$.

Step 2: Use the nonzero exponential. The exponential factor is positive and finite at every point of I . A nonzero initial Wronskian stays nonzero with the same sign. If it is zero, the initial-data columns are dependent; a nontrivial constant combination has zero value and slope at t_0 . Regular uniqueness forces that combination to vanish on I . Therefore, for solutions of this common regular equation, $W \equiv 0$ is equivalent to dependence.

Step 3: Normalize the example. Divide by $1 + t^2$, giving $p = 2t/(1 + t^2)$. Integration gives

$$\int_0^t \frac{2s}{1+s^2} ds = \ln(1+t^2), \quad W(t) = -\frac{3}{1+t^2}.$$

Thus the Wronskian is determined without finding either solution. The undivided coefficient $2t$ is not the coefficient to use in Abel's identity.

Step 4: Interpret the limits. We have $W < 0$ everywhere, with $W \rightarrow 0$ from below as $t \rightarrow \pm\infty$. The pair is independent on all of $\mathbb{R}$, and its data columns are independent at every finite time. A limiting zero outside the interval is not a zero at a point where the initial-data criterion applies.