

Question 10

For $y'' - y = 0$ on $\mathbb{R}$, fix $L > 0$. Seek solutions ℓ, r with endpoint values

$$\ell(0) = 1, \quad \ell(L) = 0, \quad r(0) = 0, \quad r(L) = 1.$$

Use $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$.

Tasks

1. Construct ℓ, r from the exponential fundamental pair and prove that they also form a fundamental set.
2. Use this pair to solve $y(0) = a, y(L) = b$ for arbitrary real a, b . Prove uniqueness.
3. For $a, b \geq 0$ and not both zero, prove positivity on $0 < t < L$ and $y(t) \leq \max(a, b)$ on $[0, L]$. Determine whether $\ell(t) + r(t) = 1$ in the interior.
4. For $a = b = 1$, locate the unique minimum on $[0, L]$ and give its value. Explain why positive interpolation weights do not imply straight-line interpolation.

Question 10 – Solution

Strategy. Choose basis functions adapted to the two measurements, then inspect their positive weights.

Step 1: Build an endpoint basis. Write $r = Ae^t + Be^{-t}$. The data give $A + B = 0$ and $A(e^L - e^{-L}) = 1$, so $A = 1/(2 \sinh L)$ and $B = -A$. Reflecting r about the midpoint gives $\ell(t) = r(L - t)$, hence

$$\ell(t) = \frac{\sinh(L - t)}{\sinh L}, \quad r(t) = \frac{\sinh t}{\sinh L}.$$

Both solve $y'' - y = 0$ and have the stated endpoint values. Their determinant at zero is $\ell(0)r'(0) - \ell'(0)r(0) = 1/\sinh L \neq 0$, so they form a fundamental set on $\mathbb{R}$.

Step 2: Fit the measurements. In this pair the endpoint coefficients are exactly the prescribed values:

$$y = a\ell + br.$$

Every solution has a unique representation, and evaluating at $0, L$ forces its coefficients to be a, b . Thus the endpoint problem has exactly one solution.

Step 3: Bound the positive weights. On $0 < t < L$, both weights are positive. The hyperbolic addition identity gives

$$\ell(t) + r(t) = \frac{\cosh(t - L/2)}{\cosh(L/2)} \leq 1 \quad (0 \leq t \leq L),$$

with strict inequality in the interior, since $|t - L/2| < L/2$. If $M = \max(a, b)$, then $0 < y$ in the interior and $y \leq M(\ell + r) \leq M$ on the full closed interval. At an endpoint the solution can equal zero if the corresponding datum is zero.

Step 4: Examine equal endpoint data. For $a = b = 1$,

$$y = \frac{\cosh(t - L/2)}{\cosh(L/2)}, \quad y_{\min} = \frac{1}{\cosh(L/2)} \text{ at } t = L/2.$$

The minimum is unique because $\cosh x$ has its unique minimum at zero. The weights are positive but sum to less than one inside; this is not the straight-line interpolation of equal endpoint values.

