

**Question 9**

For  $y'' + y = 0$  on  $\mathbb{R}$ , let the state at a time  $s$  mean the ordered pair  $(y(s), y'(s))$ . Fix a real reference time  $s$ .

**Tasks**

1. Construct a fundamental pair  $\phi_s, \psi_s$  with state data  $(1, 0)$  and  $(0, 1)$  at time  $s$ . Express any solution in that pair.
2. Derive explicit formulas for the state at  $s + h$  in terms of the state  $(a, b)$  at  $s$ .
3. Show that moving the reference time by  $h$  and then  $k$  gives the same result as moving it by  $h + k$ , and that the move by  $-h$  reverses the move by  $h$ . Prove that  $y^2 + (y')^2$  is preserved.
4. For initial state  $(3, 4)$  at zero, find the first positive time when the slope is zero and the value then. Describe the direction of travel in the state plane.

## Question 9 – Solution

**Strategy.** Changing the reference time changes the coordinates in a normalized fundamental pair, while the represented solution remains the same.

**Step 1: Normalize at the reference time.** Set

$$\phi_s(t) = \cos(t - s), \quad \psi_s(t) = \sin(t - s).$$

Both solve the equation and have the required data. Their data determinant is 1, so any solution is  $y(t) = a \cos(t - s) + b \sin(t - s)$ , where  $a = y(s)$  and  $b = y'(s)$ .

**Step 2: Move the state.** Evaluation and differentiation give

$$y(s + h) = a \cos h + b \sin h, \quad y'(s + h) = -a \sin h + b \cos h.$$

**Step 3: Compose and reverse.** Applying the same formulas with increment  $k$  yields first component

$$a(\cos h \cos k - \sin h \sin k) + b(\sin h \cos k + \cos h \sin k).$$

The addition formulas make this  $a \cos(h + k) + b \sin(h + k)$ ; the second component is  $-a \sin(h + k) + b \cos(h + k)$ . Taking  $k = -h$  gives  $(a, b)$ . Squaring the two components and adding cancels the cross terms and gives  $a^2 + b^2$ .

**Step 4: Locate the first turning point.** Here  $y = 3 \cos t + 4 \sin t$  and  $y' = -3 \sin t + 4 \cos t$ . The first positive zero of  $y'$  is

$$t_* = \arctan(4/3), \quad y(t_*) = 5.$$

Indeed  $\cos t_* = 3/5$ ,  $\sin t_* = 4/5$ , and the slope is positive before this time. In coordinates  $(y, y')$ , the state travels clockwise on the circle of radius 5: at  $(5, 0)$  its derivative is  $(0, -5)$ .

