

Question 8

Two measured solution profiles are $u(t) = t$ and $v(t) = e^{-t}$. Seek a normalized equation

$$y'' + p(t)y' + q(t)y = 0$$

with both as solutions on a connected open interval containing $t = 0$.

Tasks

1. Use the two profiles to determine p, q wherever the coefficient equations are solvable.
2. Find the largest interval containing zero on which the reconstructed coefficients are continuous, and prove that the profiles form a fundamental set there.
3. Prove that no choice of continuous normalized coefficients on an interval containing -1 can have both profiles as solutions. Does their smoothness resolve this obstruction?
4. Use the fundamental pair to solve $y(0) = 2$, $y'(0) = -1$. Verify the data and explain the distinction between extending this formula and extending the normalized equation.

Question 8 – Solution

Strategy. Treat the two known solutions as simultaneous equations for the unknown coefficients, then check the interval where the result is regular.

Step 1: Reconstruct the equation. Substituting $u = t$ and $v = e^{-t}$ gives

$$p + tq = 0, \quad 1 - p + q = 0.$$

For $t \neq -1$, these have the unique solution

$$p = \frac{t}{t+1}, \quad q = -\frac{1}{t+1}.$$

Thus both residuals vanish in $y'' + [t/(t+1)]y' - y/(t+1) = 0$ wherever these coefficients are defined.

Step 2: Find a fundamental interval. The largest connected open interval containing zero and avoiding the pole is $(-1, \infty)$. The initial-data determinant, or its value at any point of this interval, is

$$uv' - u'v = -(t+1)e^{-t} \neq 0.$$

In particular it equals -1 at zero, so the two solutions form a fundamental set throughout the interval.

Step 3: Locate the obstruction. At $t = -1$, the coefficient equations would require $p - q = 0$ and $1 - p + q = 0$, an immediate contradiction. Even finite pointwise coefficient values cannot satisfy both conditions there. Smoothness of the two functions does not guarantee a common regular normalized equation across that point.

Step 4: Solve and distinguish extensions. In $y = At + Be^{-t}$, the data give $B = 2$ and $A - B = -1$, hence

$$y = t + 2e^{-t} \quad (-1 < t < \infty).$$

The value and slope at zero are $2, -1$, and linearity verifies its equation. The formula is smooth for all real t , but the reconstructed normalized equation is undefined at -1 . Extending a function does not remove that coefficient singularity. Multiplying by $t+1$ instead gives a different, undivided formulation with a vanishing leading coefficient there.