

Question 7

Consider $y'' + 4y = 0$ on $\mathbb{R}$. Allow complex-valued solutions initially and define

$$z_+ = e^{2it}, \quad z_- = e^{-2it}, \quad i^2 = -1.$$

Then restrict attention to real-valued solutions.

Tasks

1. Verify the two complex solutions and show that they form a fundamental set over the complex numbers.
2. Find the necessary and sufficient relation between complex constants A, B for $Az_+ + Bz_-$ to be real for every real t .
3. Derive the correspondence with real coefficients in $a \cos 2t + b \sin 2t$. Explain why z_+, z_- themselves are not a fundamental set of the real-valued solution space.
4. Solve $y(0) = 1, y'(0) = -2$ in both representations and verify the coefficients.

Question 7 – Solution

Strategy. Specify the scalar field and impose real-valuedness on the entire function, not just its value at one point.

Step 1: Verify complex completeness. Each candidate has second derivative -4 times itself. Their data columns at zero are $(1, 2i)$ and $(1, -2i)$, with determinant $-4i \neq 0$. The initial-data criterion applies over $\mathbb{C}$ as well: real and imaginary parts each satisfy the real equation and its uniqueness theorem.

Step 2: Impose reality. Since $\overline{z_+} = z_-$, the conjugate of $Az_+ + Bz_-$ is $\overline{B}z_+ + \overline{A}z_-$. Independence implies equality to its conjugate exactly when

$$B = \overline{A}.$$

This relation is sufficient too, since the two terms then are conjugates.

Step 3: Convert coordinates. Euler's formula gives

$$Az_+ + Bz_- = (A + B) \cos 2t + i(A - B) \sin 2t.$$

For real a, b , the inverse relations are

$$A = (a - ib)/2, \quad B = (a + ib)/2.$$

They automatically satisfy the reality condition. The exponentials are not themselves real-valued functions on $\mathbb{R}$, so they are not elements of the real-valued solution space. A real fundamental set is $\cos 2t, \sin 2t$.

Step 4: Match the data. The real representation has $a = 1$ and $2b = -2$, hence

$$y = \cos 2t - \sin 2t = \frac{1}{2}(1 + i)e^{2it} + \frac{1}{2}(1 - i)e^{-2it}.$$

Here $A + B = 1$ and $2i(A - B) = -2$, verifying the value and slope. The conjugate coefficients ensure that the resulting solution is real everywhere.